Singularity analysis in nonlinear biomathematical models: Two case studies

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Abstract

We investigate the possession of the Painlevé Property for certain values of the parameters in two biological models. The first is a metapopulation model for two species (prey and predator) and the second one is a study of a sexually transmitted disease, into which “education” is introduced. We determine the cases for which the systems possess the Painlevé Property, in particular some of the cases for which the equations can be directly integrated. We draw conclusions for these cases.

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1. Introduction

There are four standard approaches to the analysis of systems of differential equations arising from the mathematical modelling of phenomena. They are the methods of dynamical systems, of computation, of symmetry analysis and of similarity analysis. Each method has its strengths and weaknesses.

The methods of dynamical systems provide useful qualitative information whether a system be integrable or not. Computational methods are almost invariably necessary if numerical results are required although the reliability of the numbers can be questioned if the system underlying the discretised system integrated is nonintegrable or if the method of discretisation ignores the symmetries. Symmetry and singularity analysis come into their own in the instance of integrable systems. The former provides a route to the reduction of the system of differential equations to a system of algebraic equations when a sufficient number of symmetries of the right sort are known and consequently reduces the solution of the system to a series of quadratures. Symmetry analysis is more effective when the system of differential equations contains at least one equation of second or higher order. Any system of differential equations possesses an infinite number of symmetries. Their determination is problematic without some constraint on the variable dependence in the coefficient functions. The constraint can reduce the infinity to zero even in the case of integrable systems. However, this is not the case for systems of first-order ordinary differential equations for which there exists an infinite number of point symmetries. For such systems
one must use the second part of the method of reduction of order [8,9] to introduce at least one equation of higher order. This involves the use of a nonlocal transformation and hence the potential for point/contact symmetries to be expressed as nonlocal symmetries of the original system. This is not automatically bad. It depends upon the form which the nonlocal symmetry takes [4].

By way of contrast singularity analysis is very effective in the examination of systems of first-order ordinary differential equations of polynomial or rational form. In the modelling of the evolution in ‘time’ of varied phenomena in divers fields systems of first-order ordinary differential equations which are either polynomial or rational are commonly encountered. Even in models for which arbitrary interaction or production functions can be proposed a realistic analysis invariably leads to the assumption of a polynomial or rational representation. Otherwise the analysis from the dynamical systems point of view becomes unmanageable.

In this paper, we examine two models from the viewpoint of similarity analysis. The two models are quite different in nature. The first is a ‘patch’ model due to Swihart et al. [10] in which the behaviour of a predator–prey system with different patches is treated. The relevant parts of the models are discussed in Section 2 before the analysis commences. The second is a now older study of homosexual cohorts by Hadeler and Castillo-Chavez [3] in which the effectiveness of ‘education’ is examined. The basis of the model is given in Section 3 and is then followed by the analysis.

The thrust of our treatment is the identification of integrable models. For the two systems treated we look to those values of the parameters which admit integrability. This may seem a rather arbitrary way to conduct biomathematical research, but we are encouraged so to do by the success of Torrisi and Nucci [11] with a symmetry analysis of an HIV-AIDS model. Suitable symmetries existed provided the parameters of the system were related in a particular way. This enabled the system to be reduced to quadratures. By what one could only imagine to be a curious happenchance that relationship provided a very good reflection of the numerical data available from some long-term studies in the United States [5,12]. In the instance of singularity analysis one could expect something similar. Singularity analysis selects dominant behaviour and one expects to see this after transient effects have decayed provided the nondominant terms in the system are ‘cooperative’.

In the course of our two analyses a number of points arise and we make comment upon them in the concluding section.

2. A predator–prey metapopulation model

2.1. The model

The term “metapopulation”, for a species which occupies a number of disconnected patches in space at a certain time, means the total population of the whole set of patches considered. A metapopulation model, in which both of prey and predator are present and each one is affected by the presence of the other, is presented by Swihart et al. [10]. The variables $x$ and $y$ represent a proportion of patches occupied by the prey and its specialised predator respectively. The authors start with a model by Bascompte and Solé [2], but they enrich it by introducing an ignorant predator which colonizes patches independently of whether they are occupied by the specialised prey. The equations of the dynamics are then structured as

\[
\frac{dx}{dt} = c_x(1 - x - D) - e_x(1 - y)x - (e_x + \mu)xy,
\]

\[
\frac{dy}{dt} = c_y(1 - y - D) - e_yxy - (e_y + \psi)(1 - x)y,
\]

in which the nonvariable elements are the parameters of the model. The positive colonisation terms in the cases of both the predator and the prey are the same as is the probability of colonising new patches. The parameter, $D$, is the number of patches destroyed and therefore unavailable for colonisation. There are two separate extinction probabilities for each population. The prey becomes extinct due to other natural reasons with a rate $e_x$ in the patches not occupied by the predator and with a higher rate $e_x + \mu$ in patches where the predator is present. The predator becomes extinct at a rate $e_y$ where there is prey and with a higher rate $e_y + \psi$ when it moves to patches without prey due to its ignorance. This additional term of attrition is the novel feature of the model.

By rescaling $t$, $x$ and $y$ in system (2.1) as

\[
t' = (c_x(1 - D) - e_x)t, \quad x' = \frac{c_x}{c_x(1 - D) - e_x}x, \quad y' = \frac{c_y}{c_y(1 - D) - e_y}y,
\]
(2.1) reduces to
\[
\begin{align*}
\dot{x} &= x - x^2 - bxy \\
\dot{y} &= ay - y^2 + cxy
\end{align*}
\]  
Case A,  
\begin{equation}
\tag{2.2}
\end{equation}
where
\[
\begin{align*}
b &= \frac{\mu}{c_v}, \\
a &= \frac{c_v(1 - D) - (e_v + \psi)}{c_v(1 - D) - e_s} \\
c &= \frac{\psi}{c_s}.
\end{align*}
\]

There is also the special case for which \(c_v(1 - D) - e_s = 0\). In this case the original differential equations become
\[
\begin{align*}
\dot{x} &= -c_v x^2 - \mu xy \\
\dot{y} &= (c_v(1 - D) - (e_v + \psi))y - c_v y^2 + \psi xy.
\end{align*}
\]
By the rescaling,
\[
\begin{align*}
x' &= (c_v(1 - D) - (e_v + \psi))t, \\
x &= \frac{c_s}{c_v(1 - D) - (e_v + \psi)}x, \\
y' &= \frac{c_s}{c_v(1 - D) - (e_v + \psi)}y, \\
y &= \frac{c_s}{c_v(1 - D) - (e_v + \psi)}y.
\end{align*}
\]

system (2.3) reduces to
\[
\begin{align*}
\dot{x} &= -x^2 - bxy \\
\dot{y} &= y - y^2 + cxy
\end{align*}
\]  
Case B,  
\begin{equation}
\tag{2.4}
\end{equation}
where \(b = \mu/c_v\) and \(c = \psi/c_s\).

When also
\[
c_v(1 - D) - (e_v + \psi) = 0,
\]
Eq. (2.1) becomes
\[
\begin{align*}
\dot{x} &= -x^2 - bxy \\
\dot{y} &= -y^2 + cxy
\end{align*}
\]  
Case C,  
\begin{equation}
\tag{2.5}
\end{equation}
where \(b\) and \(c\) are as in Eq. (2.4).

2.2. Case A

In order to apply the Painlevé Test we need to establish the leading order behavior of the Laurent series of the solutions. We therefore substitute into (2.2) \(x = \alpha_0^p\) and \(y = \beta_0^p\), where \(\tau = t - t_0\). By balancing the terms in (2.2) we find that \(q = p = -1\) and to leading order behaviour \(x\) and \(y\) are
\[
\begin{align*}
x &\approx \frac{1 - b}{1 + bc} \tau^{-1}, \\
y &\approx \frac{1 + c}{1 + bc} \tau^{-1},
\end{align*}
\]
where \(b \neq 1\) and \(c \neq -1\) with the latter being unaccepted as \(c\) is essentially positive.

The resonance can be found by the substitution of
\[
x = \alpha_0 \tau^{-1} + \lambda \tau^{-1} \quad \text{and} \quad y = \beta_0 \tau^{-1} + \nu \tau^{-1}
\]
into the differential equations (2.2). We find the characteristic equation
\[
\begin{pmatrix}
r - 1 + 2\alpha + b\beta & b\alpha \\
-c\beta & r - 1 - c\alpha + 2\beta
\end{pmatrix}
\begin{pmatrix}
\lambda \\
y
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
A nontrivial solution exists when the determinant of the above coefficient is zero. The roots of the characteristic equation are
\[
r = -1 \quad \text{and} \quad r = \frac{(b - 1)(c + 1)}{1 + bc},
\]
\begin{equation}
\tag{2.6}
\end{equation}
The nongeneric resonance in (2.6) is required to be an integer. With this integral value we now substitute

\[ x = \sum_{i=0}^{r} a_i t^{i-1}, \quad y = \sum_{i=0}^{r} b_i t^{i-1} \]

into the system (2.2) in order to establish the consistency conditions. The presence of nondominant terms in the equations precludes the possibility of Left Painlevé series and so we need only consider nonnegative integral values of \( r \). For \( r = 0 \) either \( b = 1 \) or \( c = 1 \), which is a contradiction to the leading order behaviour of \( q = p = -1 \). For \( r = 1 \) Eq. (2.2) sets the value of \( b = c + 2 \). When this is substituted, the consistency condition gives

\[ a = 1, \quad a_1 = \frac{-1 + (2 + c)b_1}{c}. \]

For \( r = 2 \) Eq. (2.6) gives

\[ b = \frac{3 + c}{1 - c}. \]

For positive values of \( b \) we must have \( c < 1 \). The consistency condition gives

\[ a = 1 \text{ or } c = 1 \text{ or } a = \frac{1 - c}{2}. \]

The value \( c = 1 \) is rejected since it leads to \( b \) being infinite. For \( a = 1 \) the coefficients of the resonance are related as

\[ a_2 = \frac{-1 + c + 4(3 + c)b_2}{4c(1 - c)}. \]

For \( a = (1-c)/2 \) we have

\[ a_2 = \frac{c^3 + c^2 - c - 1 + 16(c + 3)b_2}{16c(1 - c)}. \]

If we demand that \( r = 3 \), Eq. (2.6) leads to \( b = (4 + c)/(1 - 2c) \). For positive values of \( b \) we must have \( c < 1/2 \). The consistency condition gives

\[ a = 1 \text{ or } c = 1/2 \text{ or } a = \frac{4 - 5c}{8 - c} \text{ or } a = \frac{1 - 2c}{3}. \]

The value \( c = 1/2 \) is rejected since it gives an infinite value for \( b \). For \( a = 1 \), \( a = (4 - 5c)/(8 - c) \) and \( a = (1 - 2c)/3 \) the coefficients at the resonance are

\[ a_3 = \frac{4 + c}{c(1 - 2c)}b_3, \quad a_3 = \frac{4 + c}{c(1 - 2c)}b_1 \text{ and } a_3 = \frac{3(4 + c)(2 - c)}{(8 - c)^2} + \frac{4 + c}{c(1 - 2c)}b_3 \]

respectively.

For \( r = 1 \) we have \( b = c + 2 \) and \( a = 1 \) and system (2.2) becomes

\[
\begin{align*}
\dot{x} &= x - x^2 - (c + 2)xy, \\
\dot{y} &= y - y^2 + cxy.
\end{align*}
\]

If we add the two Eq. (2.7), we obtain

\[ \frac{d}{dt}(x + y) = (x + y) - (x + y)^2. \]

This is a Bernoulli equation and its solution is

\[ x + y = \frac{e^t}{A + e^t}. \]

When we substitute into Eq. (2.7b) the value of \( x \) given by (2.9), it becomes also a Bernoulli equation and its solution is

\[ y = \frac{e^t}{(A + e^t) + B(A + e^t)^{-2}}. \]

We note that the system (2.7) is a decomposed system in that it comes from the decomposition of (2.8) [1].
2.3. Case B

The leading order behavior of the Laurent series is again

\[ x \approx \frac{1 - b}{1 + bc} \tau^{-1} \quad \text{and} \quad y \approx \frac{1 + c}{1 + bc} \tau^{-1}, \]

where \( b \neq 1 \) and \( c \neq -1 \). The resonances are again

\[ r = -1, \quad r = \frac{(b - 1)(c + 1)}{1 + bc}. \]

By the same reasoning as in the previous case, we determine that there can be no resonance at \( r = 1 \) and \( 2 \). For \( r = 3 \) we have \( b = (4 + c)/(1 - 2c) \), which implies \( c = 8 \) or \( c = \frac{1}{2} \). The second case is rejected since \( b \) becomes infinite. The first case yields a negative \( b \) and this has no biological significance.

2.4. Case C

The analysis of this case is the same as Case A up to Eq. (2.6). Thereafter there is some divergence. In the first instance system C is a self-similar system as well as an autonomous system in that it possesses the symmetry \( t \partial_t - x \partial_x - y \partial_y \). All terms are dominant and so there is the potential for the existence of a Left Painlevé series as well as the Right Painlevé series of Cases A and B. (For these the existence of nondominant terms precludes the possible existence of a Left Painlevé series.)

We consider the existence of a resonance at an integer \( n \). From (2.6) the parameters, \( b \) and \( c \), are related according to

\[ b = \frac{c + n + 1}{1 + (1 - n)c}. \quad (2.11) \]

For \( n \geq 2 \) the environmental requirement that both \( b \) and \( c \) be positive demands that \( c < 1/(n - 1) \), i.e., there is an upper bound to the rate of extinction due to ignorance for which an analytic solution can occur. For \( n = 1 \) the relation (2.11) imposes no constraint. The case \( n = 0 \) is not permitted since this imposes the nonenvironmental values \( b = c = -1 \). For \( n \) a negative integer the constraint on \( c \) is that \( c > \lvert n \rvert - 1 \).

Consequently subject to the conditions stated above both Left Painlevé series and Right Painlevé series expansions are valid for Case C.

For \( r = 1 \) (\( b = c + 2 \)) system (2.4) becomes

\[ \dot{x} = -x^2 - (c + 2)xy, \]

\[ \dot{y} = -y^2 + cxy. \quad (2.12) \]

If we add the two Eq. (2.7), we obtain

\[ \frac{d}{dt}(x + y) = -(x + y)^2. \quad (2.13) \]

This is a Bernoulli equation and its solution is

\[ x + y = \frac{1}{t + A}. \quad (2.14) \]

When we substitute this into Eq. (2.12b), the value of \( x \), given by (2.14), it becomes also a Bernoulli equation and its solution is

\[ y = \frac{(t + A)^{c}}{(t + A)^{c+1} + B}. \quad (2.15) \]

We note that the system (2.12) is a decomposed system in that it comes from the decomposition of (2.13) [1].

We further note that this resonance is the only one which gives a relationship between \( b \) and \( c \) leading to a simple decomposition of the type from (2.13) to (2.12).

For negative resonances

\[ b = \frac{c + 1 - n}{(n + 1)c + 1}. \]
so that we require $c > n - 1$ for the parameters to fit the model. In this case there is a lower bound to the rate of extinc-
tion due to ignorance for which an analytic solution can occur. The lack of an upper bound adds a touch of wry amuse-
ment to the model. In the particular case that $r = -2$ we have $b = (c - 1)/(3c + 1)$. It is easily seen that for $c = 0$ or
$c = 1$ the two equations for $x$ and $y$ become Bernoulli and are easily solved. For $c = 1$ the system is
\begin{align}
\dot{x} &= -x^2, \\
\dot{y} &= -y^2 + xy.
\end{align}
(2.16)
and the solution is
\begin{align}
x &= \frac{1}{t + A} \quad \text{and} \quad y = \frac{t + A}{B(t + A)^{3/2} - 1/2}.
\end{align}
(2.17)

3. Sexually transmitted disease

3.1. The model

Hadeler and Castillo-Chavez [3] have investigated a model which examines the effectiveness of ‘education’ on the
prevalence of a sexually transmitted disease. The population is divided into two groups, the active and the inactive,
by which essentially is meant the level of variation of partners although Hadeler and Castillo-Chavez do include other
possible factors. The population is $P = A + C$, where $C$, i.e., the active, is described as the core group and $A$ as the non-
core, or inactive, group. A further subdivision is made of the core group as $C = S + V + I$, where $S$ represents the sus-
ceptibles, $V$ the ‘educated’ and $I$ the infected.

The basic model is [3] [equations (1a–d)]
\begin{align}
\dot{A} &= b(P - I) + \tilde{b}I - Ar(I, C) - \mu A, \\
\dot{S} &= Ar(I, C) - \beta SI/C - \psi S + z(1 - \gamma)I - \mu S, \\
\dot{V} &= \psi S - \beta VT/C + z\gamma I - \mu V, \\
\dot{I} &= (\beta SI + \beta VT)/C - zI - \mu I,
\end{align}
(3.1)
where $b$ and $\tilde{b}$, $\mu$ and $\mu$, represent the reproductive rates, respectively death rates, of uninfected and infected cohorts and
$\beta$ and $\tilde{\beta}$ the transmission rates from infected to susceptibles and ‘educated’. The recovery rate is $z$ and those recovering
may move to the susceptibles $(1 - \gamma)$ or the ‘educated’ $(\gamma)$. The incidence rate is proportional to $C^{-1}$ rather than $P^{-1}$
which is quite reasonable given the assumption of the noninvolvement of the noncore population, $A$. The recruitment
function, $r(I, C)$, is considered to be a function of the infected and the core group. Hadeler and Castillo-Chavez remark
that ‘Special cases are $r(I)$, where the absolute number of noninfected is observed, and $r(I/C)$, where recruitment depends on
the proportion of infected in the core’. This is a little curious as the recruitment is from $A$ to $S$ and one would expect
that an $I$-dependent recruitment rate would be from $A$ to $I$. Given the subsequent assumptions made by Hadeler and
Castillo-Chavez this point makes no difference to their subsequent analysis.

The first assumption is that this sexually transmitted disease is no more fertile/fatal than the usual vicissitudes of life
so that $\tilde{b} = b$ and $\tilde{\mu} = \mu$. Obviously HIV-AIDS is removed as one of the possible diseases, but one could consider some
sexually transmitted disease such as gonorrhoea, for which an interesting treatment has been given by Murray [7]. The
second assumption is that the two basic populations remain separate, i.e., there is no recruitment from $A$ into $C$, so that
the equations become those for an isolated core population. They are
\begin{align}
\dot{S} &= \mu C - \beta SI/C - \psi S + z(1 - \gamma)I - \mu S, \\
\dot{V} &= \psi S - \beta VT/C + z\gamma I - \mu V, \\
\dot{I} &= (\beta SI + \beta VT)/C - zI - \mu I,
\end{align}
(3.2)
We observe that (3.2) is a decomposed system [6,1], i.e., it is derived from the simple first-order ordinary differential
equation
\begin{align}
\dot{C} &= 0
\end{align}
which is obvious when the three components of (3.2) are added and $C = S + V + I$ is used.
Since $C$ is a constant and this is one of the advantages of decomposed systems in that they give an integral for the set of equations, the system (3.2) can be written as a system of two first-order ordinary differential equations, videlicet
\[
\begin{align*}
\dot{S} &= \mu - \beta SI + \alpha (1 - \gamma) I - (\mu + \psi) S, \\
\dot{I} &= (\beta - \tilde{\beta}) SI - \beta I^2 - (\alpha + \tilde{\mu} - \tilde{\beta}) I,
\end{align*}
\] (3.3)
in which the normalisation $C = 1$, i.e., $S + V + I = 1$, has been applied.

System (3.3) has a plethora of parameters which we reduce by translation and rescaling of the variables. We write
\[
\begin{align*}
S &= \frac{l + w}{C_0} b S + \frac{l}{C_0} (1 - \tilde{b}) S, \\
I &= \frac{b}{C_0} \tilde{b} I, \\
T &= \frac{t}{C_0}
\end{align*}
\] (3.3a)
so that (3.3) becomes
\[
\begin{align*}
\dot{x} &= \frac{a}{\beta - \tilde{\beta}} x + \frac{b}{\mu + \psi} x y, \\
\dot{y} &= cy + y^2
\end{align*}
\] Case A,
where
\[
\begin{align*}
a &= \frac{\beta - \tilde{\beta}}{\beta (\mu + \psi)} \frac{\alpha (1 - \gamma) (\mu + \psi) - (\mu + \psi)}{\beta - \tilde{\beta}}, \\
b &= \frac{\beta}{\beta}, \\
c &= \frac{\beta}{\mu + \psi} \frac{(\beta - \tilde{\beta})(\mu + \psi) + (\beta - \tilde{\beta}) \mu}{(\mu + \psi)^2}
\end{align*}
\] (3.4)
in the general case. We identify two special cases, videlicet for $\beta = \tilde{\beta}$
\[
\begin{align*}
\dot{x} &= -x + ay - bxy \\
\dot{y} &= cy - y^2
\end{align*}
\] Case B,
where $c$ is the same as in the previous equation with $\beta = \tilde{\beta}$.

For $\tilde{\beta} = 0$
\[
\begin{align*}
\dot{x} &= -x + ay - xy \\
\dot{y} &= cy + xy
\end{align*}
\] Case C,
where
\[
a = \frac{1}{(\mu + \psi)^2} \frac{\alpha (1 - \gamma) (\mu + \psi) - (\mu + \psi)}{\beta - \tilde{\beta}}
\] (3.5)
and $c$ is the same as in (3.6) with $\tilde{b} = 0$.

3.2. Case A

We substitute $x = x^p$ and $y = \beta \tau^q$, where $\tau = t - t_{0}$, into the dominant terms of (3.5a) and (3.5b) to determine the leading order behaviours
\[
\begin{align*}
x &\approx \left(\frac{1}{\beta} - 1\right) \tau^{-1}, \\
y &\approx \frac{1}{\beta} \tau^{-1}.
\end{align*}
\] (3.6)
To determine the resonance we substitute
\[
\begin{align*}
x &= x \tau^{-1} + \lambda \tau^{-1}, \\
y &= \beta \tau^{-1} + \nu \tau^{-1},
\end{align*}
\]
with $\alpha$ and $\beta$ as in (3.9), into the dominant terms of (3.5) to obtain the characteristic equation
\[
\begin{pmatrix}
r - 1 + \beta \beta \\
-\beta
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\nu
\end{pmatrix} = 0
\]
for which a nontrivial solution exists if
\[ r^2 + \frac{r}{b} - 1 + \frac{1}{b} = 0, \]

i.e., \( r = -1, 1 - 1/b \). The nongeneric resonance is determined by the value of the parameter.

Since not all terms in (3.5) are dominant, we substitute
\[
x = \sum_{i=0}^{r} a_i t^{i-1}, \quad y = \sum_{i=0}^{r} b_i t^{i-1}
\]
into the full (3.5) to determine the conditions for the system to be consistent. The presence of nondominant terms in (3.5) precludes the possibility of a Left Painlevé series and so we need to consider only nonnegative values of \( r \). There cannot be a resonance of 0 or 1 since the former means \( b = 1 \) and \( a = 0 \), which is in contradiction with the hypothesis of leading order behaviour, and the latter is that \( b \) is infinite. We find the first few results to be
\[
r = 2: \quad b = -1, \quad a = 2(c + 1), \quad a_2 = -\frac{1}{4} c^2 - b_2, \\
r = 3: \quad b = -\frac{1}{2}, \quad a = \begin{cases} \frac{3}{2} (c + 1) \\ \frac{1}{2} (c + 5) \end{cases}, \quad a_1 = \begin{cases} -\frac{1}{2} b_3 \\ \frac{1}{2} (c - 1 - b_3). \end{cases}
\]
The corresponding systems are
\[
\dot{x} = -x + 2(c + 1)y + xy, \\
\dot{y} = cy + xy - y^2,
\]
\[
\dot{x} = -x + \frac{3}{2} (c + 1)y + \frac{1}{2} xy, \\
\dot{y} = cy + xy - y^2,
\]
\[
\dot{x} = -x + \frac{1}{2} (c + 5)y + \frac{1}{2} xy, \\
\dot{y} = cy + xy - y^2.
\]

3.3. Case B

Since (3.7b) is free of \( x \), the equation, which is of Bernoulli’s form, can be integrated. When the solution, \( y(t) \), is substituted into (3.7a), the resulting equation is linear in \( x \). Specifically we have
\[
x = e^{-t}(A + e^{ct})^{-1} \left\{ B + \frac{c}{1 + e^{(c+1)t}} \right\}, \\
y = \frac{ce^{ct}}{A + e^{ct}},
\]
where \( A \) and \( B \) are the constants of integration.

There is no necessity to perform any analysis as the solution is given explicitly.

3.4. Case C

The leading order behaviour is \( x \approx -\tau^{-1} \) and \( y \approx \tau^{-1} \). Now the resonances are fixed at \( r = \pm 1 \) and the requirement of consistency makes \( a = -(c + 1) \). Thus the situation of \( \beta = 0 \), which reflects an absence of ‘interactions’ between the infected and ‘educated’ classes, has just the one possible value for its nongeneric resonance.

Specifically we have
\[
\dot{x} = -x - (c + 1)y - xy, \\
\dot{y} = cy + xy,
\]
so that
\[
\dot{x} + \dot{y} = -(x + y), \\
\Rightarrow \quad y = Ae^{t} - x.
\]
The resulting nonlinear first-order ordinary differential equation for \( y \) is of Bernoulli form and is trivially integrable. This is not a very physical case as it requires that \( \beta = 0 \). We include this case simply for completeness and note that it also is an example of a decomposed system.

### 3.5. A different parameterisation

In addition to the rescaling used in (3.4) we can make a variation by choosing \( B = (\mu + \psi)/\beta \) so that (3.3) becomes
\[
\begin{align*}
\dot{x} &= -x + ay - xy, \\
\dot{y} &= cy + xy - by^2.
\end{align*}
\]  

(3.17)

The leading order behaviour is
\[
x = (b - 1)r^{-1}, \quad y = r^{-1}
\]

(3.18)

and the resonances are at \( r = -1, 1-b \).

Due to (3.18) we cannot have \( b = 1 \). However, we can have \( r = 1 \) for which \( b = 0 \). Consistency requires that \( a = -c - 1 \) so that (3.17) is now
\[
\begin{align*}
\dot{x} &= -x - (c + 1)y - xy, \\
\dot{y} &= cy + xy.
\end{align*}
\]  

(3.19)

This case is the same as in (3.15).

### 4. Conclusion

We studied two separate cases of biomathematical models. The first is a metapopulation model for two species (prey and predator) presented by Swihart et al. [10]. The novel feature in this model is that the predator is ignorant and colonizes patches that may not be occupied by its prey. We investigated the possession of the Painlevé Property for this model and found that it exists for certain values of the parameters. We especially found that Eq. (2.7) can be integrated and its solution can be found explicitly. For this case the asymptotic behaviour of the predator, \( y \), for \( t \to \infty \) tends to one, which corresponds to occupation of a certain portion of the available patches. Since, as can be seen from (2.9), \( x + y \) tends also to one, the prey \( x \) tends to zero, i.e., to extinction (see Fig. 1).

There is another special case for the predator–prey system. In this case the predator–prey system is (2.12) and for the long term behavior we have that, for the parameter \( c > 0 \), both the predator and the prey become extinct (see Eqs. (2.14) and (2.15)). Another interesting aspect is that for a short period of time the predator occupies most of the sites, but finally it also becomes extinct. This behaviour is displayed in Fig. 2.

The solved case C for the predator–prey system and negative resonance \( r = -2 \) and for \( c = 1 \) show that both predator and prey become extinct, i.e., both \( x \) and \( y \) tend to zero as \( t \) tends to infinity (see Fig. 3).

![Fig. 1. The behaviour of the populations of prey and predator with respect to time for the system (2.7) in the case that \( r = 1 \) and \( c = 2 \).](image-url)
Fig. 2. The behaviour of the populations of the prey and predator with respect to time for the system (2.12) in the case that $r = 1$ and $c = 2$.

Fig. 3. The behaviour of the populations of the prey and predator with respect to time for the system (2.16) in the case that $r = -2$ and $c = 1$.

Fig. 4. The behaviour of the susceptible and the infected subgroups with respect to time as described by Eq. (3.7) of Case B.
The second system studied models a sexually transmitted disease. The model was introduced by Hadeler and Castillo-Chavez [3]. They investigate the effect of ‘education’ on the prevalence of a sexually transmitted disease. In the case $\beta = \tilde{\beta}$, for which the equations can be explicitly solved, the solution (3.14) tends for $t \to \infty$ to the equilibrium $x = c/(1 + c^4)$, $y = c$, which corresponds to constant values of $S$, $I$ and $V$, depending on the various parameters (see Fig. 4).

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References