Invariant relations in Boussinesq-type equations

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Abstract
A wide class of partial differential equations have at least three conservation laws that remain invariant for certain solutions of them and especially for solitary wave solutions. These conservation laws can be considered as the energy, pseudomomentum and mass integrals of these solutions. We investigate the invariant relation between the energy and the pseudomomentum for solitary waves in two Boussinesq-type equations that come from the theory of elasticity and lattice models.

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1. Introduction
Hydrodynamics, physics of elastic media and other fields of physics use models that after some manipulations lead to a certain class of fourth-order partial differential equations that are called Boussinesq-type equations. Boussinesq [1] originally derived an equation for a flow with free surface, but in shallow water. He reduced the original \((2+1)D\) case to \((1+1)D\) by considering an approximate solution of the former case with the assumption that the ratio of the scale of motion over the depth of the layer attains large values. The model was subsequently improved by Korteweg and de Vries [2], by considering strong surface tension (see also [3–6]).

An interesting dynamical behaviour is exhibited during martensitic–ferroelastic transformations in alloys. During the transition to the martensitic phase there appear certain twin formations and other strain structures. These can be seen as the mechanism that leads to the martensitic phase [7–12]. If one considers the microscopic level, i.e. the lattice dynamics, then one can retrieve the complex dynamics that appear at the mesoscale and explain the above formations [13–18]. In this work we are going to study a partial differential equation that models the displacement of the atoms after deformation which was introduced by one of the authors [15–18]. This model depends on the interaction of first-order neighbours and also three body interaction. The resulting differential difference equation is not manageable. Therefore, one uses the continuous approximation and arrives at a \((2+1)D\) equation. If we consider that the displacement depends only on the \(x\) coordinate then we arrive at a Boussinesq-type equation. Numerical results of the \((2+1)D\) equation starting with the solitary wave of the specific \((1+1)D\) equation as initial conditions models the strain structures of the phase transition to martensitic state in alloys.

At the macroscopic level in elasticity one uses the constitutive equation for the relationship between the stress and the strain tensors. This depends on the model used. It is inserted in the equation for the conservation of momentum and...
models the displacement. In this case also one arrives at a Boussinesq-type equation, if one considers that the constitutive equation depends not only on the strain, but also on the second gradient of the strain. This analysis can produce the second equation we are going to study which is the elastic-crystal Boussinesq equation [19–23].

The Boussinesq-type equations have two important features. First they are nonlinear equations. At the microscopic level they inherit the nonlinearity of the potential used for the displacement. At the macroscopic level the nonlinearity is included in the constitutive equation. Second they are at least fourth order with respect to the variables.

In these and various other equations the dispersion is compensated by the nonlinearity and solitary waves appear. Another important aspect of partial differential equations is their conservation laws. These are quantities that for certain solutions of the equations and especially for the solitary wave solutions studied in this work remain invariant with respect to time. There are also certain widely known partial differential equations that have an infinity of conservation laws and are completely integrable. These equations may have soliton solutions. The theory on the existence of such solutions and complete integrability for these partial differential equations was developed by Zabusky and Kruskal [24] and Gardner et al. [25]. The Boussinesq equations in [1,2] were proved to be completely integrable by Zakharov [26] and Axel and Aubry [27]. However, a partial differential equation may in general have a finite set of conservation laws without being exactly integrable. As mentioned before we are going to investigate two Boussinesq-type equations. The first is introduced in [15–18] as the continuum limit that models the formation of stain structures in phase transition of martensitic–ferroelastic type in alloys, as mentioned above. The second is the elastic-crystal Boussinesq equation [19–23]. Both of them are non-integrable in the above sense.

On the other hand, a wide class of partial differential equations are invariant under the Lie groups of time, space and function translations, i.e. they do not change their form under the action of the groups \( t \rightarrow t + \varepsilon \), \( x \rightarrow x + \varepsilon \), and \( u \rightarrow u + \varepsilon \), if \( u \) is the dependent function in the equation. We have restricted our attention to one-dimensional equations with one spatial and one temporal variable. The vector fields of these groups are \( u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \) respectively. There is also a wide class of partial differential equations that are derived from a variational principle of a Lagrangian density. If the above symmetries of such an equation, which possesses a Lagrangian representation, are also variational symmetries, then Noether’s theorem guarantees that each one of them produces a conservation law. As we will see in the main part of this work, for the pertinent equations these are the conservation laws of energy, pseudomomentum and mass.

The aim of the present work is to investigate the two specific examples of Boussinesq-type equations, mentioned above, with regard to the invariants of energy, pseudomomentum and mass. The theory was introduced by Maugin and Christov [19–23]. Section 1 contains a discussion on symmetries, variational symmetries and conservation laws of a Lagrangian system. Section 2 introduces the conserved quantities of mass, pseudomomentum and energy for the specific equations and also comments on the solitary wave solution of them. In Section 3 we evaluate these conserved quantities with the help of the properties of the partial differential equations. For the first equation we find the mass \( M_1 \), pseudomomentum \( P_1 \) and energy \( E_1 \) as functions of the velocity \( c \) of the solitary wave. We illustrate their dependence on the velocity with the help of their graphs \( M_1(c) \), \( P_1(c) \) and \( E_1(c) \) using as a parameter the rest mass \( M_{10} \). We repeat the calculations for the elastic-crystal Boussinesq equation, but in this case we are able through algebraic manipulation to find the invariant relation between the energy \( E_2 \), pseudomomentum \( P_2 \) and rest mass \( M_{20} \). In the conclusions we comment on the form of the invariant relation.

The conserved integrals of mass, pseudomomentum and energy found can be attributed to the solitary wave if considered as a point particle. The relation between the energy and the pseudomomentum is the point particle mechanics governing this fictitious particle, i.e. it can be viewed as the Hamiltonian of the point particle physics. In most cases explored by one of the authors [23] the partial differential equations studied give rise to an anti-Lorentzian type of point mechanics where the energy, the pseudomomentum and the mass tend to zero as \( c \) tends to \( c_0 \). This is also the case here, although the expression of the energy, the pseudomomentum and the mass with respect to the velocity contain also other terms than the familiar \((1 - \frac{c^2}{c_0^2})^{1/2}\).

### 2. Boussinesq equation and conservation laws

The first equation we are going to investigate is a model for the alloys that undergo a martensitic–ferroelastic phase transformation. During the transformation there appear certain twin bands and other strain modulations which can be seen as the mechanism for the transition to the martensitic phase. At the microscopic level one considers a two-dimensional lattice and also a displacement that takes place only in the \( x \) direction. The particles interact through a potential with two different kinds of terms. The first kind of terms take into account the interaction between nearest neighbours. The second kind of terms describe a three-body interaction. By using the continuum limit process the differential-difference equation that describes the motion of the particles produces a \((2 + 1)D\) partial differential equation. If we consider that
the displacement \( u \) depends only on the \( x \) coordinate then, after appropriate scaling, we arrive at the following equation [15–18]:

\[
A_1 = u_t - \frac{1}{12} u_{xxxt} - c_0^2 u_{xx} + 2u_x u_{xx} - 3u_{x}^2 + \delta u_{xxxx} = 0, \tag{1}
\]

where \( c_0^2 \) is the square of a characteristic speed and \( \delta \) is a parameter characteristic of spatial dispersion. If one uses a homogeneous extension with respect to \( y \) of the solitary waves of (1) to two spatial coordinates and evolves these initial conditions for the \((2 + 1)D\) differential equations, one gets stable formations that models the strain structures that appear at the phase transition to a martensitic state in ferroelastic alloys.

The elastic-crystal Boussinesq equation [19–23] is the second equation we will consider and is

\[
A_2 = u_t - c_0^2 u_{xx} + 2b u_x u_{xx} + d u_{xxxx} = 0 \tag{2}
\]

with \( b \) (nonlinearity parameter), \( d \) (dispersion parameter) different from zero.

It is obvious that, since (1) and (2) do not depend on \( t, x, u \), they are invariant under the Lie group action \( t \rightarrow t + \varepsilon_1 \), \( x \rightarrow x + \varepsilon_2 \) and \( u \rightarrow u + \varepsilon_3 \) respectively. The vector fields that correspond to these Lie groups are \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial u} \) and their evolutionary representatives are \( v_1 = u_t \), \( v_2 = u_x \) and \( v_3 = \frac{\partial u}{\partial u} \) respectively.

On the other hand, Eqs. (1) and (2) come from the variational principle

\[
\delta \int_{\mathbb{R}^2} L \, dx \, dt = 0, \tag{3}
\]

where the Lagrangian \( L \) for (1) is

\[
L = \frac{1}{2} u_t^2 + \frac{1}{24} t_{xx}^2 - c_0^2 \frac{1}{2} u_x^2 + \frac{1}{3} u_x^3 - \frac{1}{4} u_x^4 - \frac{\delta}{2} u_{x}^2, \tag{4}
\]

while the Lagrangian for Eq. (2) is

\[
L = \frac{1}{2} u_t^2 - c_0^2 \frac{u_x^2}{2} + \frac{b}{3} u_x^3 - \frac{d}{2} u_{xx}. \tag{5}
\]

Therefore (1) and (2) are the Euler–Lagrange equations

\[
\left( \frac{\partial L}{\partial u_t} \right)_t + \left( \frac{\partial L}{\partial u_x} \right)_x = \left( \frac{\partial L}{\partial u_{xx}} \right)_{xx} = 0
\]

with Lagrangians (4) and (5) respectively. The subscripts \( t, x \) denote partial differentiation with respect to these variables. A Lie group action is also a variational symmetry of the system if \( \int L \, dx \, dt \) remains invariant under the transformation induced by the Lie group. Since also the Lagrangians (4) and (5) that produce (1) and (2) do not depend explicitly on \( x, t, u \), the above-mentioned Lie groups with vector fields \( v_1, v_2, v_3 \) are also variational symmetries of them.

A conservation law for a differential equation is of the form

\[
\text{div} F = 0 \tag{6}
\]

for the solutions of the specific differential equations, where \( F \) depends on \( u \) and its derivatives. In our case, since we have \((1 + 1)D\) systems, it takes the form \( \partial_t F_1 + \partial_x F_2 = 0 \) where \( F = (F_1, F_2) \). The conserved quantity that remains invariant with respect to time is

\[
\int_{-\infty}^{\infty} F_1 \, dx. \tag{7}
\]

\( F_1 \) is called the density of the conservation law. The integral (7) is invariant for those solutions of the equations, for which

\[
F_2 \big|_{-\infty}^{\infty} = 0, \tag{8}
\]

since it holds that

\[
\frac{d}{dt} \int_{-\infty}^{\infty} F_1 \, dx + F_2 \big|_{-\infty}^{\infty} = 0.
\]
The solitary waves of Eqs. (1) and (2) that we are going to investigate have the property that \( u \), \( \xi \), and the higher derivatives tend to zero as \( x \to \pm \infty \).

Noether’s theorem guarantees that, if a Lie group is a variational symmetry of a differential equation, it also produces a conservation law for it. In equations like (1) and (2) there is an algorithm found through Noether’s theorem that produces the conservation law from the evolutionary infinitesimal representation of the symmetry. If \( u = Q \xi \) is the evolutionary infinitesimal representation of a variational symmetry, then

\[
Q \cdot A = \text{div} F, \tag{9}
\]

where \( A \) is the partial differential equation. We have defined \((x, t) \in \mathbb{R}^2\) since this is what we are going to need for the specific solutions. The theory holds for any open domain, but research on this subject is still open. For the above theory see e.g. [28].

3. The derivation of energy, pseudomomentum and mass

We first calculate (6) for the vector field \( v_1 \) which, since it corresponds to the homogeneity of time, i.e. to the invariance of \( A \) with respect to the Lie group \( t \to t + \xi_1 \), corresponds to the conservation of energy. For Eq. (1) we arrive at the relation

\[
(E_1)_i = \left( \frac{1}{2} u_x^2 + \frac{1}{24} u_{xx}^2 + \frac{c_0^2}{2} u_x^4 - \frac{1}{3} u_x^3 + \frac{1}{4} u_x^4 + \frac{c_0}{2} u_{xx}^2 \right)_t ,
\]

\[
= \left( \frac{1}{12} u_\xi u_{\xi \xi} + c_0^2 u_x u_x - u_x u_{\xi \xi} + u_x^3 - \frac{3}{4} u_x^4 + \frac{c_0}{2} u_{\xi \xi} \right)_x , \tag{10}
\]

while for Eq. (2) it becomes

\[
(E_2)_i = \left( \frac{1}{2} u_x^2 + \frac{c_0^2}{2} u_x^4 - \frac{1}{3} u_x^3 - \frac{c_0}{2} u_{\xi \xi}^2 \right)_t = \left( c_0^2 u_x u_x - bu_x^3 u_x - d u_{\xi \xi \xi} u_x + du_{\xi \xi} u_{\xi \xi} \right)_x . \tag{11}
\]

where \( E_1 \) and \( E_2 \) are the conserved densities of the energies for Eqs. (1) and (2) respectively. The conserved quantities are

\[
E_i = \int_{-\infty}^{+\infty} (E)_i \ dx , \tag{12}
\]

where \( i = 1, 2 \) respectively.

We calculate (6) for the vector field \( v_2 \) by multiplying \( u \) with \( \partial_A \) as shown in Eq. (9) and we find the density of the pseudomomentum, since it corresponds to the homogeneity of space, i.e. to the invariance of \( A \) with respect to space translations.

\[
(P_1)_i = \left( -u_x u_{\xi} + \frac{1}{12} u_\xi u_{\xi \xi} \right)_t = \left( -\frac{1}{2} u_x^2 + \frac{1}{24} u_{xx}^2 + \frac{c_0^2}{2} u_x^4 - \frac{2}{3} u_x^3 + \frac{1}{4} u_x^4 + \frac{c_0}{2} u_{xx}^2 - u_x^3 - \frac{3}{4} u_x^4 + \frac{c_0}{2} u_{xx} \right)_x . \tag{13}
\]

Using the same procedure for Eq. (2) we find for the conservation of pseudomomentum

\[
(P_2)_i = \left( -u_x u_{\xi} \right)_t = \left( -\frac{1}{2} u_x^2 + \frac{c_0^2}{2} u_x^4 + \frac{2b}{3} u_x^3 + du_{\xi \xi \xi} u_x - \frac{d}{2} u_{xx} \right)_x . \tag{14}
\]

\( P_1 \) and \( P_2 \) are the conserved densities of the respective pseudomomenta. The conserved pseudomomenta are

\[
P_i = \int_{-\infty}^{+\infty} (P)_i \ dx , \tag{15}
\]

where \( i = 1, 2 \) respectively. As mentioned before there is a third conservation law that stems from function translations and is related to the mass. Its form for the first equation can be retrieved by multiplying \( v_3 \) (\( Q = 1 \)) with \( \partial_A \) and is

\[
\left( u_t - \frac{1}{12} u_{\xi \xi} \right)_t = (c_0^2 u_x - u_x^3 + u_x^3 - \frac{c_0}{2} u_{xx})_x , \tag{16}
\]

while for Eq. (2) it reads
Through this conservation law we are going to define the conserved quantity of the mass after we have introduced the solitary wave solutions of the equations.

We note again that (12), (15) and the mass are conserved quantities for those solutions of Eqs. (1) and (2) that satisfy certain limit conditions for \( x \to \pm \infty \) and specifically that the quantities that appear at the right-hand side of Eqs. (10), (11), (13), (14), (16) and (17) under the derivative with respect to \( x \) tend to zero as \( x \to \pm \infty \).

It is known that Eq. (1) possesses solitary wave solutions of the form (see [15–18])

\[
(u_\xi) = \left( c_0^2 u_x - b u_x - d u_{xxx} \right)_x. 
\]

(17)

where the constants \( S_m, p, Q \) are

\[
\begin{align*}
   c^2 &= c_0^2 - \frac{S_m S_0}{2}, \\
   p &= 1 - \frac{S_m}{S_0}, \\
   Q^2 &= \frac{S_m S_0}{8r}, \\
   \gamma &= \frac{\delta - c^2}{12},
\end{align*}
\]

(19)

where \( S_0 = \frac{1}{2} - S_m \), while \( \xi = x - ct \) or \( \xi = x + ct \). Our solitary wave \( u(\xi) \) is a kink solitary wave. The condition for the existence of the above solution is \( (c_0^2 - c^2), (\delta - c^2/12) > 0 \), and \( c_0^2 - c^2 < 2/9 \); For more details on the behaviour of the solitary wave see [16,17]. Notice that the above solitary wave is a subsonic wave.

The above solution is found by considering a wave solution of (1) \( u(\xi) = u(x - ct) \) or \( u(\xi) = u(x + ct) \). We substitute such a solution in (1), integrate it once and replace \( u_\xi \) by \( f \). Then we obtain the equation

\[
\gamma f_{\xi\xi} - (c_0^2 - c^2)f + f^2 - f^3 = 0. 
\]

(20)

Notice that we put the constant of integration on the right-hand side of (20) equal to zero, since \( u_\xi \) and its derivatives tend to zero as \( x \to \pm \infty \) in order for the quantities \( E_1, P_1 \), and the mass to be conserved. This can be easily checked, i.e. that \( u_\xi, u_{\xi\xi} \) and \( u_{\xi\xi\xi} \) tend to zero as \( \xi \to \pm \infty \), from expression (18) for the solitary wave. Therefore the solitary wave solutions fall into the category of solutions of (1) for which \( E_1(\infty) = 0 \) for all the conservation laws of \( E_1, P_1 \) and the mass.

Eq. (20) can be put into Hamilton form, where \( H = T + V \) where \( T = 1/2 j^2 \) and where the potential is

\[
V = -\frac{c_0^2 - c^2}{2\gamma} j^2 + \frac{1}{3\gamma} j^3 - \frac{1}{4\gamma} j^4.
\]

Therefore (20) becomes

\[
\dot{j} = p_j, \quad \dot{p}_j = -\frac{dV}{dj},
\]

(21)

where the dot means differentiation with respect to \( \xi \) while \( p_j \) is the generalised momentum i.e. \( p_j = \frac{\partial f}{\partial j} \). Solution (18) is the homoclinic solution of Eq. (21), that tends to zero as \( x \to \pm \infty \). Zero is an unstable equilibrium of (21).

It is also known that Eq. (2) possesses solitary wave solutions of the form

\[
u_\xi = \frac{3(c_0^2 - c^2)}{2b} \cosh^2 \left( \frac{x}{2D}(\xi - \xi(0)) \right),
\]

(22)

where \( D = (c_0^2 - c^2)/d > 0 \) and where \( \xi = x - ct \) or \( \xi = x + ct \). This solitary wave \( u(\xi) \) is also a kink solitary wave. The model introduced by Boussinesq has \( d < 0 \) and is a supersonic solitary wave. For \( d > 0 \) the wave is subsonic.

Solution (22) can be found through the same procedure as for the first equation. This solution is also the homoclinic solution of the pertinent Hamiltonian equation with potential

\[
V = -\frac{c_0^2 - c^2}{2d} j^2 + \frac{b}{3d} j^3.
\]
This homoclinic solution tends to zero as \( x \to \pm \infty \), which is also an unstable equilibrium of the Hamiltonian. We have also restrained \( D = \frac{6c - c^2}{6c} \) to be greater than zero. For \( D < 0 \) the homoclinic (and all other) solution does not tend to zero as \( x \to \pm \infty \) and the quantities \( E, P \) and the mass are not conserved.

Eqs. (1) and (20) are both nonlinear. The terms \( u_1, u_2 \) of the potential of (1), which is

\[
W = \frac{c_0^2}{2} u_1^2 - \frac{1}{3} u_1^3 + \frac{1}{4} u_1^4 + \frac{\delta}{2} u_2^2,
\]

result in the nonlinearities \( 2u_1u_2 \) and \( 3u_2^2u_1 \) of (1) and also it bequeaths the nonlinearity in (21). The difference is that (21) is an integrable system since it is a one degree of freedom Hamiltonian system (it possesses the integral of the energy) while our original system (1) is not integrable.

The same holds true for the second partial differential equation and the Hamiltonian formalism.

Since the solitary waves of both equations are kink solitary waves, a natural way to define their masses is the jump from their value at \(-\infty\) to their value at infinity, i.e.

\[
M_i = \int_{-\infty}^{+\infty} u_i \, d\xi,
\]

where \( i = 1, 2 \) respectively. Quantities (24) are conserved for both the equations since the left-hand sides of the conservation laws (16) and (17) result into the integral \( \int_{-\infty}^{+\infty} u_i \, dx \), which is \(-cM_i\), being conserved.

4. The invariant relation between \( E \) and \( P \)

For the solitary wave solution \( u(x - ct) \) of Eq. (1) the conservation law for the energy (12), \( i = 1 \) becomes

\[
E_1 = \frac{c^2 + c_0^2}{2} \int_{-\infty}^{+\infty} u_1^2 \, d\xi + \left( \frac{c_0^2}{24} + \frac{\delta}{2} \right) \int_{-\infty}^{+\infty} u_1^2 u_2^2 \, d\xi - \frac{1}{3} \int_{-\infty}^{+\infty} u_1^3 \, d\xi + \frac{1}{4} \int_{-\infty}^{+\infty} u_1^4 \, d\xi.
\]

There are two equations that the derivatives of \( u \) with respect to \( \xi \) have to satisfy in order for \( u \) to be a solution of (1). For the solution (18) the partial differential equation becomes

\[
c^2 u_{\xi\xi} \equiv \frac{c^2}{12} u_{\xi\xi\xi\xi} = c_0^2 u_{\xi\xi} - 2u_{\xi}u_{\xi\xi} + 3u_{\xi\xi}u_{\xi} - \delta u_{\xi\xi\xi\xi}.
\]

By multiplying (26) by \( u \), integrating with respect to \( \xi \) from \(-\infty\) to \(+\infty\) and using integration by parts, we discover the first equation we are going to use for the derivation of \( E_1, P_1 \) and the mass \( M_1 \), that is

\[
\int_{-\infty}^{+\infty} u_1^2 \, d\xi = (c^2 - c_0^2) \int_{-\infty}^{+\infty} u_1^2 \, d\xi - \frac{12\delta - c^2}{12} \int_{-\infty}^{+\infty} u_1^2 u_2^2 \, d\xi + \int_{-\infty}^{+\infty} u_1^4 \, d\xi.
\]

The second equation is derived from Eq. (10) for the energy conservation law. For a solitary wave solution we have

\[
-c(\mathcal{E}_1)_\xi = (\mathcal{E}_1)_\xi,
\]

where \( \mathcal{E}_1 \) is the expression which is differentiated with respect to \( x \) on the right-hand side of (10) and where we have replaced the differentials with respect to \( t \) and \( x \) with differentials with respect to \( \xi \). Integrating with respect to \( \xi \) once from \(-\infty\) to \( \xi \) we find

\[
-c \int_{-\infty}^{\xi} (\mathcal{E}_1)_\xi \, d\xi = \int_{-\infty}^{\xi} (\mathcal{E}_1)_\xi \, d\xi \Rightarrow -c \mathcal{E}_1(\xi) = \mathcal{E}_1(\xi),
\]

where we have used the property \( \mathcal{E}_1(-\infty) = \mathcal{E}_1(+\infty) = 0 \) for the solitary waves of (1). If we integrate the above equation from \(-\infty\) to \(+\infty\) we discover the second equation

\[
-c \int_{-\infty}^{+\infty} \mathcal{E}_1(\xi) \, d\xi = \int_{-\infty}^{+\infty} \mathcal{E}_1(\xi) \, d\xi.
\]

We replace \( \mathcal{E}_1 \) and \( \mathcal{E}_1 \) in (30) by their forms as functions of \( \xi \), use the previous equation (27) and find

\[
\int_{-\infty}^{+\infty} u_1^2 \, d\xi = 3(c_0^2 - c^2) \int_{-\infty}^{+\infty} u_1^2 \, d\xi - \frac{3}{4}(12\delta - c^2) \int_{-\infty}^{+\infty} u_2^2 \, d\xi.
\]
By substituting $\int_{-\infty}^{+\infty} u_{1}^{2}d\xi$ and $\int_{-\infty}^{+\infty} u_{1}^{3}d\xi$ by their values as given in Eqs. (27) and (31) in the energy integral (25), it attains the form

$$E_1 = c^2 \int_{-\infty}^{+\infty} u_{1}^{2}d\xi + \delta \int_{-\infty}^{+\infty} u_{1}^{3}d\xi.$$  
(32)

The pseudomomentum $P_1 = -\int_{-\infty}^{+\infty} u_{1}u_{t}d\xi$ for the solitary wave is easily seen to be

$$P_1 = c \int_{-\infty}^{+\infty} u_{1}^{2}d\xi + \frac{c}{12} \int_{-\infty}^{+\infty} u_{1}^{3}d\xi.$$  
(33)

For the subsonic wave of Eq. (18) it holds that $1 - \rho > 0$. Due to this fact it is easily computed that the integral of the mass (24) is

$$\int_{-\infty}^{+\infty} u_{1}d\xi = \frac{S_{m}}{Q} \tanh^{-1}(\sqrt{1 - \rho \tanh(Q\xi)}) \bigg|_{-\infty}^{+\infty}.  
(34)$$

By taking the limits as $\xi \to \pm\infty$, we calculate

$$M_1 = 4(2\gamma)^{1/2} \tanh^{-1}(\sqrt{1 - \rho}).$$  
(35)

If we consider the expressions of $S_{m}$, $Q$ and $\rho$ for $c = 0$ we obtain the rest mass $M_{10}$ which is given by (35) for $c = 0$ and depends only on $c_{0}^{2}$ and $\delta$. Since the dependence of $M_1$ on $c$, with $M_{10}$ as a parameter, i.e. the relation $M_1 = M_1(c, M_{10})$ cannot be calculated analytically, we present the curves $M_1 = M_1(c)$ for various values of the parameter $M_{10}$ in Figs. 1 and 2. In Fig. 1 the parameter $\delta$ takes the values $\delta = 2, 5, 10, 15, 20, 25$ successively from the bottom curve to the top one.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The curves $M_1(c)$ with varying $\delta$ and $c_{0}^{2} = 0.2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The curves $M_1(c)$ with varying $c_{0}^{2}$ and $\delta = 5$.}
\end{figure}
and \( c_0^2 = 0.2 \), while in Fig. 2 the parameter \( c_0^2 \) is \( c_0^2 = 0.1, 0.15, 0.2, 0.21, 0.22, 0.2222222 \) successively from the bottom curve to the top one and \( \delta = 5 \).

It is easily seen that \( M_1 = 0 \) for \( c^2 = c_0^2 \). It can also be calculated that \( M_{10} \rightarrow +\infty \) as \( c_0^2 \rightarrow \frac{1}{2} \).

The energy integral can also be calculated from its expression (32) if one replaces \( \int_0^{\infty} u_1^2 \, d\xi \) and \( \int_0^{\infty} u_2^2 \, d\xi \) by their values. It is found to be

\[ E_1 = c^2 A(c) + \delta B(c), \]  

(36)

where

\[ A(c) = \frac{8}{3} (2\gamma)^{1/2} \tanh^{-1}(\sqrt{e}) - 4 \gamma^{1/2} (c_0^2 - c^2)^{1/2}, \]  

(37)

and

\[ B(c) = -\frac{4}{3(2\gamma)^{1/2}} \left( \frac{4}{9} - 2(c_0^2 - c^2) \right) \tanh^{-1}(\sqrt{e}) + \frac{4(c_0^2 - c^2)^{1/2}}{9\gamma^{1/2}} - \frac{4(c_0^2 - c^2)^{3/2}}{3\gamma^{1/2}} \]  

(38)

and where \( e = 1 - p \). The curves of \( E_1 = E_1(c) \), with the same variation of parameters as for the integral of the mass, are given in Figs. 3 and 4. In these figures the parameters \( \delta \) and \( c_0^2 \) increase also from the bottom curve to the top one respectively.

It is easily computed from the above expression for the energy integral that \( E_1 = 0 \) for \( c^2 = c_0^2 \). On the other hand, it attains a finite value for \( c_0^2 = \frac{1}{2} \) as \( c \rightarrow 0 \).

Finally, the integral of the pseudomomentum is calculated from its expression (33) if one substitutes the values of \( \int_0^{\infty} u_1^2 \, d\xi \) and \( \int_0^{\infty} u_2^2 \, d\xi \). It is found to be

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Fig. 3. The curves \( E_1(c) \) with varying \( \delta \).

Fig. 4. The curves \( E_1(c) \) with varying \( c_0^2 \).
\[ P_1 = cA(c) + \frac{c}{12}B(c), \]  

(39)

where \( A(c) \) and \( B(c) \) are given by (37) and (38) respectively. The curves of \( P_1 = P_1(c) \) with the same variation of parameters as before are given in Figs. 5 and 6. In this case also, the parameters increase from the bottom curve to the top one.

It is easily shown from the above expression for the pseudomomentum integral that \( P_1 = 0 \) for \( c^2 = c_0^2 \) and that \( P_1 = 0 \) for \( c = 0 \). It also tends to zero for \( c_0^2 = \frac{1}{2} \) as \( c \to 0 \). Note that for \( c_0^2 = \frac{1}{2} \) and \( c \to 0 \) both \( E \) and \( P \) attain finite values, while \( M \) goes to infinity. Note also that the solitary wave, for \( c_0^2 = \frac{1}{2} \), degenerates into a steady state solution \( f = S_m \).

We proceed to calculate the integrals of \( E_2, P_2 \) and \( M_2 \) for the case of the second partial differential equation (2). In this case, the two equations, that are found for these integrals in the same way as in the previous equation, are adequate for finding their value without actually calculating directly the integrals. This and also their expressions with respect to \( c \) allows us to find explicitly their dependence on the rest mass \( M_{20} \) and also the invariant relation between \( E_2 \) and \( P_2 \).

For the solitary wave (22) of Eq. (2) the energy integral becomes

\[ E_2 = \frac{c^2 + c_0^2}{2} \int_{-\infty}^{+\infty} u^2 \, d\xi = \frac{b}{3} \int_{-\infty}^{+\infty} u_1^2 \, d\xi + \frac{d}{2} \int_{-\infty}^{+\infty} u_\xi^2 \, d\xi. \]  

(40)

The partial differential equation (2) also becomes

\[ (c^2 - c_0^2)u_{\xi\xi} + 2bu_\xi + du_{\xi\xi\xi\xi} = 0. \]  

(41)
By solving the system of Eqs. (42) and (43) we can find

\[(c_0^2 - c^2) \int_{-\infty}^{+\infty} u_1^2 \mathrm{d} \xi - b \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi + d \int_{-\infty}^{+\infty} u_1'' \mathrm{d} \xi = 0. \]  \hspace{1cm} (42)

The second equation in this case, analogous to (30), is found to be

\[\frac{c^2 - c_0^2}{2} \int_{-\infty}^{+\infty} u_2^2 \mathrm{d} \xi + \frac{2b}{3} \int_{-\infty}^{+\infty} u_1^3 \mathrm{d} \xi - \frac{3d}{2} \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi = 0. \]  \hspace{1cm} (43)

By solving the system of Eqs. (42) and (43) we can find \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \) and \( \int_{-\infty}^{+\infty} u_2^2 \mathrm{d} \xi \) as functions of \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \). They are

\[\frac{b}{3} \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi = \frac{c_0^2 - c^2}{5} \int_{-\infty}^{+\infty} u_1^2 \mathrm{d} \xi. \]  \hspace{1cm} (44)

and

\[d \int_{-\infty}^{+\infty} u_1'' \mathrm{d} \xi = \frac{c_0^2 - c^2}{5} \int_{-\infty}^{+\infty} u_1^2 \mathrm{d} \xi. \]  \hspace{1cm} (45)

Replacing relations (44) and (45) in (40) we find \( E_2 \) as a function of \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \).

\[E_2 = \frac{4c^2 + c_0^2}{5} \int_{-\infty}^{+\infty} u_1^2 \mathrm{d} \xi. \]  \hspace{1cm} (46)

The pseudomomentum \( P_2 = -\int_{-\infty}^{+\infty} u_1 u_2 \mathrm{d} \xi \) is easily seen to be

\[P_2 = c \int_{-\infty}^{+\infty} u_1^2 \mathrm{d} \xi. \]  \hspace{1cm} (47)

By a procedure analogous to that which resulted in relation (30) we find for the conservation law (17)

\[c \int_{-\infty}^{+\infty} \mathcal{M}_2 \mathrm{d} \xi + \int_{-\infty}^{+\infty} \mathcal{M}_2 \mathrm{d} \xi = 0, \]  \hspace{1cm} (48)

where \( \mathcal{M}_2 \) is the expression inside the differential with respect to \( t \) on the left-hand side of the conservation of mass equation (17), while \( \mathcal{M}_2 \) is the expression inside the differential with respect to \( x \) on the right-hand side of the conservation of mass equation. Eq. (48) establishes a relation between \( \int_{-\infty}^{+\infty} u_1 \mathrm{d} \xi \) and \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \) which is

\[b \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi = (c_0^2 - c^2) \int_{-\infty}^{+\infty} u_1 \mathrm{d} \xi. \]  \hspace{1cm} (49)

For the solitary waves \( u(x - ct) \) the mass is

\[M_2 = M_{20} \left(1 - \frac{c^2}{c_0^2}\right)^{1/2}, \]  \hspace{1cm} (50)

where \( M_{20} \) is the rest mass and can only be defined for the subsonic solitary wave which will be considered from now on. It is easily seen to be

\[M_{20} = \frac{6}{b}(c_0^2)^{1/2}. \]  \hspace{1cm} (51)

Eq. (49) relates \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \) with \( M_2 \) whose value is given in (50). Therefore substituting \( \int_{-\infty}^{+\infty} u_1' \mathrm{d} \xi \) in Eqs. (46) and (47) we immediately recover the expressions of \( E_2 \) and \( P_2 \) with respect to \( c \),

\[E_2 = E_{20} \left(1 + 4\frac{c^2}{c_0^2}\right) \left(1 - \frac{c^2}{c_0^2}\right)^{3/2}, \]  \hspace{1cm} (52)

\[P_2 = P_{20} c \left(1 - \frac{c^2}{c_0^2}\right)^{3/2}, \]
where

\[ E_{20} = \frac{6c_0^2}{b_0^2} d^{1/2} c_0^3 \]  

(53)

and

\[ P_{20} = \frac{6c_0^2}{b_0^2} d^{1/2} c_0^3 \]  

(54)

For \( u(x + ct) \) the energy and the mass remain the same but the pseudomomentum changes sign. Therefore there exist two solitary waves with opposite velocities which have the same energy and mass but opposite pseudomomenta.

If we replace \( (c_0^2 - c^2) \) in the expression for the energy from the expression for the pseudomomentum in (52), we find that

\[ E_2 = \frac{4c_0^2 + c^2}{5c} P_2. \]  

(55)

We solve relation (55) with respect to \( c \) and denote the two roots of this second degree polynomial by \( c_+ \) and \( c_- \). They are expressed as functions of \( E_2 \) and \( P_2 \) as

\[ c_\pm = \frac{SE_2 \pm \sqrt{E_2^2 - 16c_0^2P_2^2}}{8P_2}. \]  

(56)

Therefore, if we substitute in the value of \( P_2 \) given in relation (52) either \( c_+ \) or \( c_- \) we find the invariant relation between \( E_2 \) and \( P_2 \), i.e.

\[ P_2 - \frac{6c_+^2}{b_\pm^2} (d(c_0^2 - c_+^2))^{1/2} (c_0^2 - c_+^2) = 0 \quad \text{or} \]

\[ P_2 - \frac{6c_-^2}{b_\pm^2} (d(c_0^2 - c_-^2))^{1/2} (c_0^2 - c_-^2) = 0. \]  

(57)

In order to have the invariant relation as a rational function of \( E_2 \) and \( P_2 \) it is preferable to take the square of the functions of \( P_2 \) with respect to \( c_+ \) and \( c_- \) in Eqs. (57) and multiply them together, since either one or the other is valid. Thus we arrive at the following invariant relation between \( E_2 \) and \( P_2 \), where we have also replaced \( M_{20} \) as given by (51)

\[ P_2^2 = \frac{M_{20}^2}{b_\pm^2} \frac{5^3}{4E_{\pm}^4 P_2^4} \left( 9 \cdot 4^4 c_\pm^6 p_\pm^6 E_{\pm}^2 - 7 \cdot 4^3 5^2 c_\pm^4 E_{\pm}^2 P_2^4 + 5^4 4^2 c_\pm^2 E_{\pm}^2 P_2^2 - 5^5 E_{\pm}^8 - 2 \cdot 4^4 c_\pm^8 p_\pm^8 \right) + \frac{M_{20}^2}{b_\pm^4} \frac{5^6 c_\pm^6}{4^8} \left( c_0^2 - E_{\pm}^2 \right)^3 P_2^2 = 0. \] 

(58)

The relations for the solitary wave \( u(x + ct) \) are easily derived from the previous relations by the transformation \( c \to -c \). The invariant relation (58) obviously remains the same.

5. Conclusions

If one considers the solitary waves of the above equations as fictitious “particles” then Eqs. (35), (36) and (39) reveal the dependence of the integrals of the mass, energy and momentum of Eq. (1) on the velocity of these “particles”. In other words they reveal the point mechanics induced by Eq. (1). For Eq. (2) we are also able to compute analytically the invariant relation (58) between the energy and the momentum, i.e. the law of the mechanics for this equation. One can see that it is neither a classical (Newton or Einstein–Lorentz) theory relation nor a classical field theory one. This is more of the anti-Lorentzian type with a momentum vanishing both for zero and a finite characteristic speed (compare the anti-Lorentzian mechanics identified numerically by Christov and Maugin for generalized Boussinesq systems [21,22]). For comparison it is salient to remember that for Newtonian quasi-particles the invariant relation is \( 2M_0 E - P^2 \) (quasi-particles associated to the nonlinear Schrödinger (cubic) equation), while for relativistic quasi-particles this relation reads \( E^2 = M_0 c_0^4 + P^2 c_0^2 \) (quasi-particles associated to the sine-Gordon equation). In all cases slow (compared to \( c_0 \)) quasi-particles behave like Newtonian ones.

For our case, for the solitary wave produced by the elastic-crystal Boussinesq equation, it is easily computed, by expanding \( E_2 \) and \( P_2 \), as given in relations (52), in Taylor series with respect to \( c \) and keeping terms up to order \( c^2 \), that the approximate invariant relation found is
\[ E_2 = E_{20} + \frac{\kappa}{2} \frac{P_1^2}{M_{20}}, \]

where \( \kappa = b/c_0^2 \) and \( E_{20} = \frac{M_{20}}{\delta} c_6^0. \)

For the soliton of Eq. (1) after some tedious, but straightforward, algebra the approximate relation between \( E_1 \) and \( P_1 \), as \( c \to 0 \) is

\[ E_1 = E_{10} + \left\{ \frac{6}{B(0) + 12A(0)} - 2^5 \frac{(\delta/c_0)^{1/2}}{(B(0) + 12A(0))^{1/2}} \right\} P_1^2, \]

where \( E_{10} = \delta B(0). \)

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