Ψ-series and obstructions to integrability of periodically perturbed one degree of freedom Hamiltonians

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Abstract

A connection between the Ψ-series local expansion of the solution of a perturbed system of ODEs and the evaluation of the Mel’nikov vector with the method of residues has recently been found by Goriely and Tabor. By following an analogous procedure, we find a straightforward relation between the failure of the compatibility condition of the Painlevé test and the absence of an analytic integral for periodically perturbed Hamiltonians whose unperturbed part does not necessarily possess a homoclinic loop. We apply these results to a periodically perturbed anharmonic oscillator.

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1. Introduction

This work deals with the connection between the Ψ-series expression of the solution in complex time and a recently developed [1] real-time non-integrability criterion for one degree of freedom periodically perturbed Hamiltonian systems, whose integrable part is also Painlevé integrable. We follow the lines of reasoning of a recent paper by Goriely and Tabor [2], where they establish an analogous connection between Ψ-series of the solutions of perturbed ODEs and Mel’nikov’s integral on the homoclinic loop of the unperturbed part, when this part possesses such a loop. On the one hand, the non-integrability criterion developed in Ref. [1] involves an integral computed on the bounded solutions of the autonomous part of the Hamiltonian with a period that is in resonance with the period of the perturbation. This integral can be calculated by the method of residues. On the other hand, in the Ψ-series expansions for the solution of the perturbed system, certain coefficients of the logarithmic part are related to the same residues through the variational equations of the unperturbed system.

This integral which is evaluated on the periodic orbits of the unperturbed system is the subharmonic Mel’nikov function (e.g. Ref. [3], p. 109). This function first appeared in Ref. [4], Ch. 3, for the case of n degrees of freedom autonomous Hamiltonians and in Ref. [5] for periodically perturbed one degree of freedom systems. Simple zeroes of this function correspond to bifurcations of the non-isolated periodic orbits of the unperturbed system to the perturbed one. The non-vanishing of this integral has been linked to the non-integrability of the perturbed system both in

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the autonomous \( n \) degrees of freedom case \([6,7]\) and the periodically perturbed one \([1]\). In the autonomous case of two degrees of freedom, this integral supplies a non-integrability criterion which is equivalent to the non-integrability theorem of Poincaré. The non-vanishing of this integral has also been related to the infinite Riemann sheets of the solution of the perturbed system \([8,9]\).

Painlevé and his associates, in their work on complex second-order ODEs, developed the so-called \( \alpha \)-method (e.g. Ref. \([10]\)), in order to obtain necessary conditions for which the only movable singularities of the solutions of such equations are poles. Then by direct computation, they found all such differential equations that possess the above property, which is termed the Painlevé property. Later, Ablowitz, Ramani and Segur \([11]\) introduced a test which poses necessary conditions for the Painlevé property in complex ODEs of any order. More references on Painlevé analysis and its connection to PDEs can be found in Ref. \([2]\).

The solutions of the perturbed Hamiltonian can be expressed in series containing logarithmic terms, which belong to the class called \( \Psi \)-series and have found many applications in the theory of complex differential equations (e.g. Refs. \([10,12]\)). They have also been employed in some recent works by several authors \([13–17]\).

### 2. Laurent and \( \Psi \)-series

We consider the one degree of freedom Hamiltonian

\[
H = H_0(q, p) + \epsilon H_1(q, p, t).
\]

The perturbation \( H_1 \) is polynomial in \( q, p \), analytic and periodic with real period \( T_1 \) with respect to its explicit dependence on \( t \). The equations of motion are

\[
\dot{x} = \Omega DH(x) = \Omega DH_0 + \epsilon \Omega DH_1,
\]

where

\[
x = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \partial / \partial q \\ \partial / \partial p \end{pmatrix}.
\]

Another assumption is that the unperturbed part, which is integrable, has the Painlevé property, i.e. the only movable singularities in its solution are poles. We also suppose that its solution can be expanded in formal Laurent series

\[
x = (t - t_*)^p \sum_{k=0}^{\infty} a_k(t - t_*)^k
\]

around the poles, where \( t_* \) is the position of the pole, which contains one more arbitrary constant, in addition to the arbitrary \( t_* \).

In order to apply the Painlevé test \([11]\) to the unperturbed system, one finds first the dominant behaviour of the solution around any existing pole, i.e.

\[
x \sim \alpha(t - t_*)^p,
\]

where

\[
\alpha = a_0 \in \mathbb{C}^2 \setminus \{0\}, \quad p \in \mathbb{Z}^2.
\]

By balancing the dominant terms, the unperturbed part of \((2)\) takes the form

\[
\dot{x} = \Omega DH_0(x) = \Omega D\tilde{H}_0 + \Omega D\tilde{H}_1,
\]

where \( \Omega D\tilde{H}_0 \) is the part of \( \Omega DH_0 \) that exhibits the dominant behaviour \((t - t_*)^{p-1}\), while the behaviour of \( \Omega D\tilde{H}_0 \) is \((t - t_*)^{p+1-1}, p \in \mathbb{Z}^2\setminus\{0\}\). There may be different integer vectors \( p \) that balance Eq. \((4)\) and all of them have to be taken into account. Non-integer (rational or irrational) powers \( p \) imply that the system, although integrable, does not comply with the requirements of the Painlevé property. Since the unperturbed equations are Painlevé integrable by assumption, they pass the first step of the test with integer \( p \) as dominant behaviour for all balances.

The coefficients \( a_j \) are determined by a recursive procedure through the equation

\[
(R - jl)a_j = -P_j(\alpha, a_1, \ldots, a_{j-1}),
\]

where \( R = \Omega D^2\tilde{H}_0(\alpha) - \text{diag } p, \Omega D^2\tilde{H}_0 \) is the Hessian of \( \tilde{H}_0 \) and \( P_j \) is a polynomial vector which depends only on \( \alpha, \ldots, a_{j-1} \).

The position of the resonances, that is the coefficients of the Laurent series where the arbitrary constant enters in the expansion, is determined by the eigenvalues \( r \) of the matrix \( R \), since for these eigenvalues the determinant of the matrix on the left-hand side of Eq. \((5)\) becomes zero and therefore the corresponding solutions \( a_r \) involve arbitrary constants. In
order for the $a_r$ to exist, $P_r$ must be orthogonal to the eigenvector $\hat{\beta}_r$ of $R^T$ of the eigenvalue $r$, i.e.
\[ \hat{\beta}_r^T P_r = 0. \] (6)

The eigenvalue $r_1 = -1$ always appears, since it corresponds to the arbitrariness of the position of the pole. If the unperturbed system possesses the Painlevé property, the other eigenvalue is a positive integer and the compatibility condition (6) is satisfied.

In the Painlevé integrable case, the assumptions that the system has the Painlevé property and that there exist formal Laurent solutions which contain the arbitrary constants, that can be checked directly by the Painlevé test, suffice for the convergence of these Laurent series in a domain around the pole. The above-mentioned assumptions also suffice for the non-existence of a natural boundary and fixed essential singularities.

We are also going to deal with periodic orbits of the unperturbed system, with period $T_2$ (in real time), that are in resonance with the external period $T_1$, i.e.
\[ T = nT_1 = mT_2, \] (7)
where $m, n \in \mathbb{Z} \setminus \{0\}$ and are relative primes. The frequency of the unperturbed orbits is
\[ \omega = \frac{dH_0}{dj} = \frac{2\pi}{T_2}, \]
where $j$ is the action variable of $H_0$. We assume that the unperturbed system is non-degenerate, i.e. that
\[ \frac{d^2H_0}{dj^2} \neq 0, \]
holds in an open domain of phase space. As a consequence, the period $T_2$ of the solutions varies continuously in this domain, and in this case a dense set of periodic orbits of $H_0$ satisfying the resonance condition (7) for different $m, n$ exists. We also assume that these periodic solutions possess, in complex time, a second, complex period $T_2$, such that $\text{Im}(T_2) \neq 0$.

We also assume that the term $DH_1$ is non-dominant, i.e. the dominant behaviour of this term is \((t - \tau_*)^{p+1} \) with $\tau \in \mathbb{C} \setminus \{0\}$.

If one tries to find the Laurent expansion of the perturbed solution
\[ x' = \sum_{j=0}^{\infty} a_j (t - \tau_*)^{p+j}, \]
the non-dominant behaviour of the perturbation leads to $a_0' = \alpha$ while the corresponding equation (5) for the perturbed system takes the form
\[ (R - j\ell)a'_j = -P'_j(\alpha, a'_1, \ldots, a'_{j-1}), \] (8)
where the matrix $R$ is the same as in (5). Therefore, the position $r \neq -1$ of the arbitrary coefficient remains the same, but now in general the compatibility condition (6) is not satisfied, i.e.
\[ \hat{\beta}_r^T P'_r \neq 0, \] (9)

since the perturbed system is expected not only to violate the Painlevé property, but also to be non-integrable. Finally, the last assumption allows us to use for $\varepsilon \neq 0$ an expansion containing logarithmic terms which is termed logarithmic $\Psi$-series.

**Proposition 1** [2]. The solutions of the perturbed system can be expanded in the formal $\Psi$-series
\[ x' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ijk} (t - \tau_*)^{p+i+1} \varepsilon^j Z^k, \] (10)
or, alternatively
\[ x' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_{ij} \varepsilon^j Z^i, \] (11)
where $Z = \log(t - \tau_*)$ and $s_{ij}$ are Laurent series that converge in a complex domain around the pole of the unperturbed system.

For a proof of this proposition see Ref. [2].

3. An obstruction to integrability and its evaluation with residues

In Ref. [1] we have proved the following:

**Proposition 2** [1]. A one degree of freedom perturbed Hamiltonian of the form $H = H_0(x) + \varepsilon H_1(x, t)$ where the perturbation is periodic in time with period $T_1$ and $H_0$ is non-degenerate, does not possess an integral of motion, analytic in $\varepsilon$ in an open interval around zero if, in an open domain of the
phase space, one can find a dense or a key set of periodic orbits (invariant circles) of $H_0$ with period $T_2$ for which the resonance relation (7) holds, such that

$$I = \int_{t_0}^{t_0+T} \frac{\partial H_1}{\partial t} (x(t - t_0), t) \, dt \neq 0$$

for at least one $t_0$ for each orbit.

Let $[H_0, H_1] = (DH_0)^T Q DH_1$ be the standard Poisson bracket. Then the above obstruction can also be expressed as

$$I = \int [H_0, H_1] \, dt \neq 0$$

for this set of resonant periodic orbits of $H_0$, since

$$\int_{t_0}^{t_0+T} \frac{\partial H_1}{\partial t} \, dt = \int_{t_0}^{t_0+T} [H_1, H_0] \, dt + \int_{t_0}^{t_0+T} \frac{\partial H_1}{\partial t} \, dt \equiv 0.$$  \hspace{1cm} (14)

The above calculations are performed in real time and the above criterion guarantees that no real-analytic and single-valued integral of the perturbed system exists in the domain where the above-mentioned set of periodic orbits is dense, or in an open domain determined by a key set of them. We may put without loss of generality $T_1 = 2\pi$ by a scale transformation in time, so that the resonance relation (7) becomes $T = 2\pi n = mT_2$.

Let us now evaluate the integral (13) by the method of residues. First we evaluate the Poisson bracket $[H_0, H_1]$ along the periodic solutions of the unperturbed part. Since $H_0$ is autonomous, $x$ depends on time only through $\tau = t - t_0$, while $H_1$ depends on time both explicitly and through the unperturbed solution $x(\tau)$, i.e.

$$[H_0, H_1]\{x(t - t_0), t\} = [H_0, H_1]\{x(\tau), \tau + t_0\}.$$  \hspace{1cm} (15)

Since the Poisson bracket is, as shown above, a periodic function with period $2\pi$ with respect to its second argument, we can expand it in Fourier series as

$$[H_0, H_1] = \sum_{k=-\infty}^{\infty} g^{(k)}(x(\tau)) e^{ik(\tau + t_0)}.$$  \hspace{1cm} (16)

Then the integral (13) becomes

$$I = \sum_{k=-\infty}^{\infty} I^{(k)} e^{ikt_0}, \quad I^{(k)} = \int_0^T g^{(k)}(x) e^{ikt} \, d\tau,$$

i.e. $I$ is periodic in $t_0$ with period $2\pi$. The coefficients $I^{(k)}$ are going to be determined, for $k \neq 0$, with the help of the following path integrals,

$$S^{(k)} = \oint_\gamma g^{(k)}(x(z)) e^{ikz} \, dz,$$

where $\gamma$ is the path in the complex plane depicted in Fig. 1 and can be decomposed as $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

The relationship between $I^{(k)}$ and $S^{(k)}$ is the following,

$$S^{(k)} = I^{(k)} + S_2^{(k)} + S_3^{(k)} + S_4^{(k)},$$  \hspace{1cm} (16)

where

$$S_i^{(k)} = \oint_\gamma g^{(k)}(x(z)) e^{ikz} \, dz, \quad i = 2, 3, 4.$$

It can be easily shown that $S_2^{(k)} = -S_4^{(k)}$ and $S_3^{(k)} = -e^{ikt_0} I^{(k)}$, therefore by (16)

$$I^{(k)} = \frac{1}{1 - e^{ikt_0}} S^{(k)}.$$  \hspace{1cm} (17)

The value of the path integral $S^{(k)}$ is given by

$$S^{(k)} = 2\pi i \sum_{\nu} \text{res} \{g^{(k)}(x(\tau))e^{ikt}\},$$  \hspace{1cm} (18)

where the summation is performed over the residues of the function $g^{(k)}(x(\tau))$ on all the poles of the unperturbed solution $x(\tau)$, that are included in the previous parallelogram, which is different for each resonant periodic orbit.

The above procedure is not valid for $k = 0$ but, since the function $\partial H_1/\partial t$ does not have a zeroth-order
Fourier coefficient in an expansion in $\tau + t_0$ with respect to its explicit dependence on time and since (14) is valid, i.e., $I^{(k)}$ is also the $k$th Fourier coefficient of (12), $I^{(0)} = 0$ holds.

If a number of poles lie on the period lines $\gamma_2, \gamma_4$, then the path taken in order to perform the integration can be selected in a convenient way around them, such that the previous results are still valid. There are no poles on the period lines $\gamma_1, \gamma_3$ since we have assumed that $H_0$ is analytic in its arguments and therefore so is the real solution $x(t)$.

4. Variational equations of the unperturbed system and the Laurent expansion of their solutions

By expanding around a periodic solution $x_0(t)$ of system (2), we obtain for the first variation $\xi$ the linear system of variational equations

$$\dot{\xi} = \Omega D^2 H_0 \xi, \quad (19)$$

where $D^2 H_0$ is evaluated along the solution $x_0(t)$. It is known (Ref. [4], § 64) that $\Omega D H_0(x_0(t))$ is a periodic solution of (19).

In the following we are also going to use the adjoint variational equations (e.g. Ref. [18], p. 87) of (19),

$$\dot{\bar{\xi}} = -\bar{\xi}^T \Omega D^2 H_0(x_0(t)) \quad (20)$$

It is easy to show that, if $\xi = \Omega y$ is a solution of (19), then $y^T$ is a solution of (20). Let $Y, \bar{Y} \in GL(2, C(t))$ be fundamental solutions of (19) and (20), respectively, i.e. $\dot{Y} = \Omega D^2 H_0 Y$ and $\dot{\bar{Y}} = -\bar{Y} \Omega D^2 H_0$. If $Y$ is a solution of (19), then $Y^{-1}$ is a solution of (20) (e.g. Ref. [2]).

As a next step, we are going to use the following proposition.

Proposition 3 [2]. The system of variational equations (19) around a periodic solution of the unperturbed system has a fundamental local solution $Y$, such that each column $y^{(i)}$, $i = 1, 2$, has a convergent local Laurent expansion in a punctured disk $B(t_*, y^{(i)})$ around each pole $t_*$ of the solution $x_0(t)$. The form of the column $y^{(i)}$ around $t_*$ is

$$y^{(i)}(t) = (t - t_*)^{p_i} \left( \beta_{p_i} + \sum_{j=1}^{\infty} b_j^{(i)} (t - t_*)^j \right),$$

where $\beta_{p_i}$ is the eigenvector of $R$, corresponding to the eigenvalue $\alpha_{p_i}$. The adjoint variational equations (20) have also a fundamental local solution $\bar{Y}$, whose rows $\bar{y}^{(i)}$ have the form

$$\bar{y}^{(i)}(t) = (t - t_*)^{p_i - r_i - 1} \left( \bar{\beta}_{p_i} + \sum_{j=1}^{\infty} \bar{b}_j^{(i)} (t - t_*)^j \right)$$

in a neighbourhood $B(t_*, \bar{y}^{(i)})$ around $t_*$, where $\bar{\beta}_{p_i}$ is the eigenvector of $R^T$, of eigenvalue $\alpha_{p_i}$. The matrices $Y, \bar{Y}$ are defined in $B(t_*, Y) = \bigcap_{i=1}^{2} B(t_*, y^{(i)})$ and $B(t_*, \bar{Y}) = \bigcap_{i=1}^{2} B(t_*, \bar{y}^{(i)})$, respectively. In the open domain $B(t_*, Y) \cap B(t_*, \bar{Y})$ we have $\bar{Y} = Y^{-1}$.

For a proof, see Ref. [2]. For the Hamiltonian case at hand, we will prove the following:

Proposition 4. $DH_0$, evaluated at $x_0$, is a solution of the adjoint variational equations with dominant behaviour $-p - r$ where $r$ is the other (in addition to $-1$) eigenvalue of $R$. Moreover, the coefficient at the dominant behaviour is an eigenvector of $R^T$ with eigenvalue $-r$ and therefore its local expansion can be selected as a row of the fundamental solution $Y$.

Proof. Since the leading behaviour of $x_0(t)$ is $x_0 \sim \alpha_i (t - t_*)^{p_i}$, the leading behaviour of $\Omega D H_0$ is $p - 1$, since $\Omega D H_0(x_0) = x_0 \sim \alpha_i p_i (t - t_*)^{p_i - 1}$. By inserting now $\Omega D H_0$ in the variational equations, we obtain

$$\alpha_i p_i (p_i - 1) \sim \sum_j \Omega \frac{\partial^2 \bar{H}_0}{\partial x_i \partial x_j}(\alpha) \alpha_j p_j$$

or

$$R a p = -\alpha p, \quad (21)$$

i.e. $-1$ is an eigenvalue of $R$ and the coefficient of $\Omega D H_0$ at the dominant behaviour $p - 1$, i.e. $\alpha = (\alpha_1 p_1, \alpha_2 p_2)$ is the corresponding eigenvector.

As mentioned above, $DH_0(x_0)^T$ is a solution of the adjoint variational equations (20). Let $q$ be its dominant behaviour. From the identity $DH_0 = \Omega^{-1}(\Omega D H_0)$ we see that

$$q_1 = p_2 - 1, \quad q_2 = p_1 - 1. \quad (22)$$

Since $\Omega D^2 H_0$ is traceless, the trace of $R$ equals $-(p_1 + p_2)$. Let $r$ be the eigenvalue of $R$, different from $-1$. 


Then \( r - 1 = -p_1 - p_2 \) and the leading behaviour of \( DH_0 \) is also
\[
q_1 = -p_1 - r, \quad q_2 = -p_2 - r.
\] (23)

As seen from (22) and (23), \( \Omega \text{ diag } p \Omega^{-1} - 1 = -\text{diag } p - rl \) holds, so that
\[
(R^T - rl) \Omega^{-1} \alpha p = \Omega(R + l) \alpha p = 0,
\]
i.e. the vector of the coefficients of the dominant behaviour of \( DH_0 \), which is \( \Omega^{-1} \alpha p = (-\alpha_2 p_2, \alpha_1 p_1) \), is an eigenvector of \( R^T \) of eigenvalue \( r \). \( \Box \)

So we conclude that the adjoint variational equations (20) have a local fundamental solution \( \tilde{V} \), which can always be selected as
\[
\tilde{V} = \left( \begin{array}{c}
\tilde{y}^{(1)T} \\
(DH_0)\tilde{y}
\end{array} \right) \in \text{GL}(2, C(t)).
\] (24)
The dominant behaviour of \( \tilde{y}^{(1)} \) is, according to Proposition 3, \(-p + 1 \).

5. Relation of the compatibility condition to the non-integrability of the perturbed system

By truncating the formal logarithmic expansion of the solution of the perturbed system around a pole \( t_* \) of the unperturbed one to order \( \epsilon \), Eq. (11) yields
\[
x' = s_{00} + \epsilon(s_{10} + s_{11} Z) + O(\epsilon^2).
\]
On the other hand, if we consider the solution \( x' \) as a perturbation of the unperturbed solution \( x \), i.e. \( x' = x + \epsilon x_1 + O(\epsilon^2) \) and differentiate with respect to time, we get up to order \( \epsilon \)
\[
\dot{x}_1 = \Omega D^2 H_0 x_1 + \Omega DH_1,
\]
where \( D^2 H_0 \) and \( DH_1 \) are evaluated on the unperturbed solution \( x \). Since \( x_1 = s_{10} + s_{11} Z \), taking into account that the polynomial and the logarithmic functions are linearly independent, we arrive at the following system,
\[
\dot{s}_{10} = \Omega D^2 H_0 s_{10} + \Omega DH_1 - s_{11}(t - t_*)^{-1},
\]
\[
\dot{s}_{11} = \Omega D^2 H_0 s_{11}.
\] (26)

By assuming that the logarithmic term enters at \( O(\epsilon) \) in the expansion of \( x' \), i.e. \( s_{11} \neq 0 \), the following proposition holds.

Proposition 5 [2]. The solution of the system (26) can be written as
\[
s_{10} = YK_0, \quad s_{11} = YK_1,
\] (27)
where
\[
K_0 = \oint_{t_*} \tilde{V}(s) \Omega DH_1(x(s), s) \, ds - K_1 \log(t - t_*),
\] (28)
\[
K_1 = \text{res}_{t_*} \{ \tilde{V}(t) \Omega DH_1(x(t), t) \}
\] (29)
and \( t_* \) is the pole around which we expand the solution \( x = s_{00} \) of the unperturbed and \( x' \) of the perturbed system.

The proof, which can be found in Ref. [2], involves the general solution (27) of (26), while (29) is obtained by demanding \( s_{10} \) to be a Laurent series.

Now, if we select the matrix \( \tilde{V} \) as in (24), for the components of \( K_1 \) we have
\[
K_{11} = \text{res}_{t_*} \{ \tilde{y}^{(1)T} \Omega DH_1 \},
\]
\[
K_{12} = \text{res}_{t_*} \{ [H_0, H_1] \},
\] (30)
while from (27) we obtain \( K_1 = \tilde{V}s_{11} \), i.e.
\[
K_{11} = \tilde{y}^{(1)T}s_{11}, \quad K_{12} = DH_0^T s_{11}.
\] (31)

According to Eqs. (10) and (11) of Proposition 1, \( s_{11} = \sum_{i=0}^{\infty} a_i(t - t_*)^{i+p} \) and, according to Proposition 4,
\[
\tilde{y}^{(1)} = (t - t_*)^{-p + 1} \left( \tilde{\beta}_{-1} + \sum_{j=1}^{\infty} b_j^{(1)}(t - t_*)^j \right),
\]
\[
\tilde{D}H_0 = (t - t_*)^{-p - r} \left( \tilde{\beta}_r + \sum_{j=1}^{\infty} b_j^{(2)}(t - t_*)^j \right),
\]
where \( \tilde{\beta}_{-1} \) and \( \tilde{\beta}_r \) are eigenvectors of \( R^T \) of eigenvalues \( -1 \) and \( r \), respectively. Now Eq. (31), taking into account that \( K_1 = \text{const.} \) yields
\[
K_{11} = 0, \quad K_{12} = \tilde{\beta}_r^T a_{11} \overset{\text{def}}{=} c_r.
\] (32)
Considering now, following Ref. [2], the logarithmic expansion of \( x' \) up to \( p + r \) and expanding with respect to \( \epsilon \) we obtain
\[ x' = \sum_{i=1}^{r} \left( a_i \bigg|_{e=0} + e \frac{\partial a_i}{\partial e} \bigg|_{e=0} \right) (t - t_*)^{p+i} \]
\[ + \left( b_0 \bigg|_{e=0} + e \frac{\partial b_0}{\partial e} \bigg|_{e=0} \right) (t - t_*)^{p+r} Z + \text{h.o.t.} \]

Since there are no logarithmic terms in the unperturbed expansion, \( b_0 \bigg|_{e=0} = 0 \) holds and the coefficient of the term \( e \log(t - t_*) \) at \( p + r = \partial b_0 / \partial e \bigg|_{e=0} \), which equals \( a_{r+1} \), according to (10).

Introducing now the logarithmic term, Eq. (8) becomes \((R - rI)a'_r = -P'_r + b_0 \) and, in order for \( a'_r \) to exist, the right-hand side must be orthogonal to the eigenvector \( \beta_r \), i.e.
\[ \beta_r^T P'_r = \beta_r^T b_0 \overset{\text{def}}{=} C_r, \tag{33} \]
therefore, according to (32), we have
\[ K_{12} = \beta_r^T a_{r+1} = \beta_r^T \frac{\partial b_0}{\partial e} \bigg|_{e=0} = \frac{\partial C_r}{\partial e} \bigg|_{e=0} = c_r. \tag{34} \]

Inserting into the second equation of (30) the expansion (15) and taking into account (34) we get
\[ K_{12} = \sum_{k=0}^{\infty} \text{Res} \{g^{(k)}(x(\tau))e^{ik\theta} \} e^{ik\theta} \]
\[ = \sum_{k=0}^{\infty} c_r^{(k)} e^{ik\theta} = c_r. \tag{35} \]

Taking now into account Eqs. (17), (18) and (35), we obtain
\[ I^{(k)} = \frac{2\pi i}{1 - e^{ikT_c}} \sum_{r} c_r^{(k)} \tag{36} \]
and the integral \( I \) in the obstruction (12) or (13) equals
\[ I = 2\pi i \sum_{k=0}^{\infty} \frac{\sigma_r^{(k)}}{1 - e^{ikT_c}} e^{ik\theta}, \]
\[ \sigma_r^{(k)} = \sum_{r} c_r^{(k)}, \quad k \neq 0, \tag{37} \]
where \( \sigma_r^{(k)} \) is the sum of the Fourier coefficients \( c_r^{(k)} \) of the terms of order \( e \) of the compatibility condition (33) over all poles of the solution \( x(\tau) \) of the unperturbed system inside the parallelogram depicted in Fig. 1. Since \( e^{ik\theta} \) are linearly independent functions, from (37) we see that if, for at least one \( k \), the sums \( \sigma_r^{(k)} \) are different from zero for the invariant circle of \( H_0 \) obeying the resonance condition (7), then \( I \neq 0 \) on this circle. By combining this result with Proposition 2, we have proved the following proposition.

**Proposition 6.** If at least one \( \sigma_r^{(k)} \) is different from zero on a dense or a key set of resonant invariant circles of \( H_0 \), the perturbed Hamiltonian does not possess an analytic integral of motion for \( e \neq 0 \) in an open domain around zero.

### 6. Application to a periodically perturbed anharmonic oscillator

We will apply the above results to the perturbed anharmonic oscillator defined by the Hamiltonian
\[ H = H_0 + eH_1 = \frac{1}{2} p^2 + \frac{1}{4} x^4 + \varepsilon x \cos t, \tag{38} \]
The solution of the unperturbed system is (e.g. Ref. [19], p. 207)
\[ x(\tau) = \lambda \text{cn}(\Lambda \tau, 1/\sqrt{2}), \quad \tau = t - t_0, \tag{39} \]
where \( \lambda \) and \( t_0 \) are arbitrary constants. The periods of \( x(\tau) \) are (e.g. Ref. [20], p. 914)
\[ T_2 = \frac{4K}{\lambda}, \quad T_c = \frac{2(K + iK')}{\lambda}, \]
where \( K \) and \( K' \) are respectively the complete and the complementary complete elliptic integral of the first kind with modulus \( k = 1/\sqrt{2} \).

The resonance condition (7) takes the form
\[ \frac{4K}{\lambda} = \frac{2\pi n}{m}, \tag{40} \]
where \( m, n \) are relative prime positive integers. For the invariant circle of \( H_0 \) defined by (40), the path of integration \( \gamma \) is the parallelogram shown in Fig. 1, in which \( 2m \) simple poles of the solution (39) are included, at the points
\[ t_*^k = \frac{1}{\lambda} [2(2s + 1)K + iK'], \]
\[ \tilde{t}_*^k = \frac{1}{\lambda} (4sK + iK') \tag{41} \]
for \( s = 0, 1, \ldots, m - 1 \), with residues \( i/k = i\sqrt{2} \) and \(-i/k = -i\sqrt{2}\), respectively.

Applying the standard Painlevé test to the equations of motion, we obtain for the leading behaviour the following results: \( p = (-1, -2), \ \alpha = \pm(i\sqrt{2}, -i\sqrt{2}), \)

where the plus sign corresponds to the expansion around \( r^*_+ \) and the minus sign to the one around \( r^- \).

The matrix \( R \) is

\[
R = \Omega D^2 \dot{H}_0(\alpha) - \text{diag } p = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}
\]

and the eigenvalues are \(-1\) and \(r = 4\). By expanding \( \Delta \dot{H}_0 \) we obtain the normalized eigenvectors

\[
\bar{p}_4 = \mp i\sqrt{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

The vector \( P'_4 \), on the other hand, is

\[
P'_4 = \begin{pmatrix} 0 \\ -\varepsilon \sin(t + t_0) \end{pmatrix},
\]

so that \( c_4 = \pm i\sqrt{2} \sin(t + t_0) \). The corresponding \( c^{(k)}_r \) are non-zero only for \( k = \pm 1 \) and are given by

\[
c^{(-1)}_4(r^*_+) = \frac{1}{\sqrt{2}} e^{-ir^*}, \quad c^{(1)}_4(r^*_+) = \frac{1}{\sqrt{2}} e^{ir^*},
\]

\[
c^{(-1)}_4(r^-) = \frac{1}{\sqrt{2}} e^{-ir^-}, \quad c^{(1)}_4(r^-) = -\frac{1}{\sqrt{2}} e^{ir^-}.
\]

We apply (36) for \( k = \pm 1 \) and distinguish the following cases:

(a) \( m \neq 1 \);

(b) \( m = 1, n = 2j \): In these cases \( I^{(-1)} = I^{(1)} = 0 \) so that \( I = 0 \).

(c) \( m = 1, n = 2j + 1 \): In this case (38) yields

\[
I^{(-1)} = -\frac{2\pi}{K} \frac{q^{-(j+1/2)}}{1 + q^{-(2j+1)}},
\]

\[
I^{(1)} = \frac{2\pi}{K} \frac{q^{(j+1/2)}}{1 + q^{(2j+1)}},
\]

where \( q = e^{-\pi} \) is the elliptic nome and we obtain

\[
I = \frac{-4\pi}{K} \frac{q^{(j+1/2)}}{1 + q^{(2j+1)}} \sin t_0.
\]

This integral is not identically zero, so the obstruction of Proposition 2 is satisfied on the set of invariant circles of \( H_0 \) defined by

\[
\lambda = \frac{2K}{\pi(2j + 1)}, \quad j = 0, 1, \ldots,
\]

which is given from (40) for the above values of \( m \), \( n \). These circles for \( j \to \infty \) accumulate on the equilibrium \( x = p_x = 0 \) and they form a key set in an open neighbourhood of it. According to Proposition 2, the system does not possess an analytic integral of motion for \( \varepsilon \neq 0 \). This result is valid, however, for all \( \varepsilon \), since the above analysis remains unaltered if we transform \( \varepsilon \to C \varepsilon \) with arbitrary \( C \).

7. Conclusions

The main result of this paper is included in Proposition 6, which relates in a straightforward way the compatibility condition of the Painlevé test to the non-integrability of a periodically perturbed nearly integrable Hamiltonian of one degree of freedom. In obtaining this result, we followed the main steps of the method developed by Goriely and Tabor in Ref. [2], where they established such a relation for the homoclinic Mel'nikov vector of a perturbed system of ODEs whose integrable part possesses a homoclinic loop. Our result connects the \( \Psi \)-series expansion of the perturbed solution directly to the non-existence of an analytic integral, irrespectively of whether the unperturbed system possesses such a loop or not.

We have shown that if the sum \( \sigma^{(k)}_r \) of the compatibility conditions at order \( \varepsilon \) over all poles of the local expansions for a dense or a key set of periodic orbits of \( H_0 \) is different from zero, then the perturbed Hamiltonian cannot possess an analytic integral for \( \varepsilon \neq 0 \). This does not mean, however, that the introduction of logarithmic terms in the local expansion around each pole at order \( \varepsilon \) leads to the non-existence of an analytic integral, since \( c^{(k)}_r \) may be in general different from zero, but their sums may eventually vanish. In this case, the obstruction to integrability of Proposition 2 is not satisfied and one cannot prove non-integrability. This is exactly the case for most of the invariant circles of the integrable part in the example of the preceding section, although (12) is satisfied in a key set of them.

The present analysis combines also the singularity structure of the perturbed solutions to the subharmonic bifurcations of periodic orbits, since the integral \( I \), as commented in Ref. [1], is actually the subharmonic Mel'nikov integral, whose simple zeroes define the
points of continuation of periodic orbits of the unperturbed to the perturbed system. The possible existence of such a relation was already pointed out in Ref. [2]. As we mentioned in Ref. [1], the obstruction (12) poses a weaker condition than the existence of simple zeroes of I.

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References