A criterion for non-integrability based on Poincaré's theorem

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The non-integrability of a two-degrees-of-freedom Hamiltonian is investigated by a method based on a well-known theorem of Poincaré. It is proved that the perturbed system is non-integrable for values of the perturbative parameter in an open interval around zero, if, on a dense set of resonant tori of the unperturbed system, the average value of the perturbative function, evaluated along the periodic orbits on each torus, depends on the particular orbit. An application to the separable quartic oscillator with a quadratic perturbation is made and it is shown that if the perturbation is non-separable in the same coordinates, the system is non-integrable.

1. Introduction

One of the most important qualitative features of the motion in a Hamiltonian system is its integrability. According to the famous Liouville–Arnold theorem, a Hamiltonian system of $n$ degrees of freedom is integrable if it possesses $n$ independent integrals of motion in involution. In this case, a transformation to local “action-angle” variables exists and the bounded motions are windings on invariant tori. Though one may prove integrability of a two-degrees-of-freedom Hamiltonian by obtaining a second invariant of motion, the proof of its non-integrability is a rather difficult task, even though the latter is a generic property of Hamiltonian systems.

One of the methods for proving non-integrability relies on proving the existence of transverse homoclinic (or heteroclinic) points in the associated Poincaré map. This property implies non-integrability according to a theorem by Moser [1]. For Hamiltonian systems of two degrees of freedom such theorems have already been supplied by Poincaré ([2] p. 397) and for the one-degree-of-freedom non-autonomous case by Melnikov [3]. Related theorems can also be found in [4,5], while applications of this method to particular systems have been made by several authors [6–10].

Ziglin [11] introduced an important theorem on non-integrability, by studying the branching of solutions of an integrable system in the complex domain. By applying Ziglin’s theorem to homogeneous potentials, Yoshida [12] offered a simple algorithm which may prove non-integrability of such potentials and can also be applied to potentials that are sums of homogeneous terms [13]. A generalization of Ziglin’s theorem has been given by Ito [14], while applications of Ziglin’s or Yoshida’s theorems have been made in [15–20]. For the geometric aspects of Ziglin’s theorem, see [21–23].

At the end of the previous century, Poincaré ([24] p. 233) in his pioneering work on the three-body problem supplied a theorem for proving non-integrability of a two-degrees-of-freedom Hamiltonian of the form $H_0 + \epsilon H_1$, where $H_0$ is an integrable Hamiltonian and $\epsilon$ a small
parameter, while according to another theorem of Poincaré ([24] p. 184) a Hamiltonian is non-integrable if its isolated periodic orbits form a dense set in phase space.

In this paper, by proceeding along the lines of the proof of the first theorem of Poincaré mentioned above, we prove that a two-degrees-of-freedom Hamiltonian $H = H_0 + \varepsilon H_1$, where $H_0$ is a non-degenerate integrable part in which action-angle variables can be defined in an open domain of the phase space, is non-integrable for values of $\varepsilon \neq 0$ in an open interval around zero, if, for a dense set of resonant tori of $H_0$, the average value of $H_1$, evaluated along the periodic orbits on each one of them, depends on the particular orbit, i.e. it is different in at least two of them.

2. The theorem of Poincaré

Let us consider the perturbed, autonomous, two-degrees-of-freedom Hamiltonian

$$H = H_0 + \varepsilon H_1, \quad (1)$$

where $H_0$ is the integrable part. Let us further assume that $H$ is integrable, possessing an integral $\Phi$, independent of $H$ and analytic in $\varepsilon$ around $\varepsilon = 0$, i.e. it can be expanded in an infinite sum of powers of $\varepsilon$,

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + O(\varepsilon^2). \quad (2)$$

Since $H$ and $\Phi$ are in involution, we obtain at zeroth order of $\varepsilon$

$$[\Phi_0, H_0] = 0, \quad (3)$$

that is, $\Phi_0$ is an integral of $H_0$, and at first order

$$[\Phi_0, H_1] + [\Phi_1, H_0] = 0. \quad (4)$$

Equations (3) and (4) hold identically in phase space. As is shown in [24], $\Phi_0$ cannot depend only on $H_0$, since, if this were the case, we could construct another integral $\Phi'$ of $H$ which starts with another $\Phi_0'$ that is not solely a function of $H_0$. Otherwise we would end up with a $\Phi$ dependent on $H$, a contradiction to the original assumption.

Let $I_0$ be a known independent integral of $H_0$. Then we can define action-angle variables $J_1, J_2, w_1, w_2$ in the domain of bounded motion, where the gradients of $H_0$, $I_0$ are non-collinear. It is then known [24] that $\Phi_0$ does not depend on the angles $w_i$ in an open domain of the phase space, if the non-degeneracy condition

$$\det \left( \frac{\partial^2 H_0}{\partial J_i \partial J_j} \right) \neq 0 \quad (5)$$

holds in this domain.

At this stage, in order to prove Poincaré's theorem, one expands both $H_1, \Phi_1$ as doubly periodic Fourier series in $w_1, w_2$ and, since eq. (4) is an identity in phase space, one obtains the infinite equations

$$a_{m_1, m_2} \left( m_1 \frac{\partial \Phi_0}{\partial J_1} + m_2 \frac{\partial \Phi_0}{\partial J_2} \right) = \beta_{m_1, m_2} (m_1 \omega_i + m_2 \omega_i), \quad (6)$$

where $m_1, m_2 \in \mathbb{Z}$,

$$a_{m_1, m_2}, \quad \beta_{m_1, m_2}$$

are the coefficients of the multiple Fourier expansions of $H_1$ and $\Phi_1$ respectively, and

$$\omega_i = \frac{\partial H_0}{\partial J_i} (i = 1, 2)$$

are the frequencies of the unperturbed system. The right-hand side of eq. (6) becomes zero on a dense set of points, for various $m_1, m_2$, in any open domain where (5) holds. At these points, the equation

$$m \omega_1 - n \omega_2 = 0 \quad (7)$$

is satisfied for $m, n$ relative primes, and such that $m/n = -m_1/m_2$. This means that either

$$m \frac{\partial \Phi_0}{\partial J_1} - n \frac{\partial \Phi_0}{\partial J_2} = 0 \quad (8)$$

or
\( a_{m_1,m_2} = 0 \)

is true at the dense set of points where (7) is valid. If at least one of the above coefficients of a class defined by \( m_1/m_2 = -m/n \) is different from zero at the corresponding points for all such classes, then eqs. (7) and (8) are simultaneously true on a dense set of points, which mean that the relation

\[
\frac{D(\Phi_0, H_0)}{D(J_1, J_2)} = 0
\]

is an identity on the whole domain, i.e. \( \Phi_0 = \Phi_0(H_0) \). This is a contradiction to the assumption that there exists a second independent integral \( \Phi \) of \( H \), analytic in \( \varepsilon \), so there exists an open interval of \( \varepsilon \) around zero, where the perturbed system for \( \varepsilon \neq 0 \) is non-integrable.

3. A criterion for non-integrability

We are now going to deviate from the above procedure in order to obtain the criterion presented in the introduction. By a suitable parametrization, eq. (4) can also be realized in the form

\[
\frac{d\Phi}{dt} = [H_1, \Phi_0], \tag{9}
\]

where both sides of (9) are calculated on a particular solution of the unperturbed Hamiltonian \( H_0 \). On a member of the family of resonant tori \( \omega_1/\omega_2 = n/m \ (m, n \in \mathbb{Z}\setminus\{0\}) \) of \( H_0 \), all the solutions are \( T \)-periodic, where \( T \) is determined by

\[
T = \frac{2\pi}{\Omega}, \quad \Omega = \frac{\omega_1}{n} = \frac{\omega_2}{m}, \tag{10}
\]

and thus \( \Phi_1 \), when calculated on any of these periodic solutions, must be a periodic function of time with the same period, i.e.

\[
\Phi_1(T) = \Phi_1(0). \tag{11}
\]

By integrating (9) along a periodic solution and taking into account (11) and the fact that the \( \frac{\partial \Phi_0}{\partial J_j} \) depend only on the actions, we obtain the condition

\[
\frac{\partial \Phi_0}{\partial J_1} \int_0^T \frac{\partial H_1}{\partial w_1} dt + \frac{\partial \Phi_0}{\partial J_2} \int_0^T \frac{\partial H_1}{\partial w_2} dt = 0. \tag{12}
\]

Notice that the two integrals in (12), divided by \( T \), correspond to the constant coefficients of the complex Fourier series of their integrands, considered as periodic functions of the time along the particular periodic solution. In order to determine these coefficients, we consider the double Fourier expansion of \( H_1 \) with respect to the angles,

\[
H_1 = a_{0,0} + \sum_{k_1,k_2} \alpha_{k_1,k_2} \exp[i(k_1w_1 + k_2w_2)], \tag{13}
\]

where the coefficients in (13) depend only on the actions. Differentiating (13) we obtain

\[
\frac{\partial H_1}{\partial w_j} = i \sum_{k_1,k_2} k_j \alpha_{k_1,k_2} \exp[i(k_1w_1 + k_2w_2)]
\]

\((j = 1, 2).\)

The unperturbed solution on the particular torus is

\[
w_1 = \omega_1 t + \vartheta_1, \quad w_2 = \omega_2 t + \vartheta_2, \tag{14}
\]

where the \( w_i \) are mod \( 2\pi \) and the \( \vartheta_i \) are arbitrary constants. Equations (10) and (14) result to

\[
k_1w_1 + k_2w_2 = (k_1n + k_2m)\Omega t + (k_1\vartheta_1 + k_2\vartheta_2), \tag{15}
\]

so that the constant coefficient of the Fourier series of \( H_1 \) with respect to time, along a periodic orbit on the resonant torus \((n/m)\) is

\[
\langle H_1 \rangle = a_{0,0} + \sum_{k=-\infty}^{\infty} \sum_{m} \alpha_{mk,-nk} \exp(ik\vartheta), \tag{16}
\]

where

\[
\vartheta = m\vartheta_1 - n\vartheta_2.
\]

The infinite sum in (16) originates from the
coefficients of (13) for which \( k_1 n + k_2 m = 0 \).
Note that on the certain resonant torus \( (n/m) \) there exists a one-to-one correspondence between a particular orbit and \( \theta \) (mod \( 2\pi \)). In a similar way we find

\[
\int_0^T \frac{\partial H_1}{\partial w_1} \, dt = \imath m T \sum_{k \neq 0} k a_{mk,-nk} \exp(ik \theta),
\]

and eq. (12) transforms to

\[
\left( m \frac{\partial \Phi_0}{\partial J_1} - n \frac{\partial \Phi_0}{\partial J_2} \right) \sum_{k \neq 0} k a_{mk,-nk} \exp(ik \theta) = 0.
\]

Thus the periodicity condition on \( \Phi_1 \), on an unperturbed resonant torus translates to

\[
\frac{\partial \Phi_0}{\partial J_1} - \frac{\partial \Phi_0}{\partial J_2} = 0
\]

(18a)
or

\[
\sum_{k \neq 0} k a_{mk,-nk} \exp(ik \theta) = 0
\]

(18b)
on every orbit on the torus. Let us investigate conditions (18) a bit more. If (18a) holds for one orbit on the torus, it holds for every orbit on this torus, since the \( \partial \Phi_0/\partial J \) depend only on \( J_1, J_2 \) that remain constant on it. On the other hand, since (18b) depends on \( \theta \) explicitly, it may be true for every orbit on the torus or only for some of them. If (18b) holds for every such orbit, there is nothing to prevent \( \Phi_1 \) from existing and the assumption meets no contradiction at this step, but, since \( \theta \) is arbitrary and \( \exp(ik \theta) \) are independent functions, it also means that

\[
a_{mk,-nk} = 0 \quad \forall k \in \mathbb{Z}\setminus\{0\}.
\]

Therefore, if at least one coefficient \( a_{mk,-nk} \) is non-zero, there exists at least one orbit on the resonant torus, for which the left-hand side of (18b) is non-zero. Obviously, the inverse is true as well, i.e. if all coefficients \( a_{mk,-nk} \) are zero, then the left-hand side of (18b) is zero for every \( \theta \).

The above analysis leads us to the consideration of the infinite sum

\[
\sum_{k \neq 0} a_{mk,-nk} \exp(ik \theta).
\]

which appears in the expression (16) of the constant coefficient of the Fourier series of \( H_1 \) with respect to time along a periodic orbit. If we find at least one \( \theta \) (which defines one orbit on the torus) for which the sum (19) is non-zero, there exists at least one coefficient which is different from zero. This result, if it is true for a specific perturbative function \( H_1 \), in combination with the previous analysis, certifies that there exists at least one periodic orbit on the torus on which (18b) does not hold and consequently (18a) must be true on the whole torus. According to eq. (16), proving that (19) is non-zero is equivalent to proving that

\[
\left\langle H_1 \right\rangle = \frac{1}{T} \int_0^T H_1(t) \, dt
\]

is not equal to \( \alpha_{\theta,0} \), that is \( \left\langle H_1 \right\rangle \) is not a constant on the particular resonant torus, i.e.

\[
\frac{d\left\langle H_1 \right\rangle}{d\theta} \neq 0
\]

for at least one orbit.

Let us suppose now that a dense set of resonant tori has been found, such that on each one of them the average value of \( H_1 \) along the periodic orbits of \( H_0 \) is not a constant but varies from orbit to orbit. Then, for at least one orbit on each torus, (18b) does not hold and thus, in order for \( \Phi_1 \) to exist, (18a) must hold on the dense set of tori, on each one for the corresponding integers \( m, n \). As explained in the preceding section, this would mean that \( \Phi_0 \) depends on \( H_0 \) in an open domain of the phase space, which is a contradiction to the original assumption. Thus for the perturbed system no
integral $\Phi$, analytic in $\varepsilon$ and independent of $H$, exists in this domain. So at this point we may state the following:

**Proposition.** Consider the two-degrees-of-freedom autonomous Hamiltonian $H = H_0 + \varepsilon H_1$, where $H_0$ is a non-degenerate integrable part for which we can define action-angle variables at least in an open domain of phase space. If there exists a set of resonant tori of $H_0$, which is dense in the action space, such that on each one of them the average value of the perturbative function $H_1$, evaluated on the corresponding periodic orbits, depends on the particular orbit, there exists an interval $(-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ of $\varepsilon$ in which the perturbed Hamiltonian $H$ is non-integrable.

4. **An application to the quartic oscillator**

We will now apply the results of the preceding section to the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(x^4 + y^4) + \varepsilon(\alpha x^2 + \beta y^2 + 2xy),$$

where $\alpha$, $\beta$ are constant parameters. The integrable part

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(x^4 + y^4)$$

(23)

corresponds to a separable unharmonic oscillator and the general solution of the equations of motion for $H_0$ is given by ([25], p. 207)

$$x = \lambda \cn(\lambda t - \varphi_1, 1/\sqrt{2}),$$

$$y = \mu \cn(\mu t - \varphi_2, 1/\sqrt{2}),$$

(24)

while

$$p_x = \dot{x}, \quad p_y = \dot{y}$$

and where $\lambda$, $\mu$, $\varphi_1$, $\varphi_2$ are arbitrary constants which depend on the initial conditions. Two independent integrals of motion for $H_0$ are the following:

$$I_1 = 2p_x^2 + x^4 = \lambda^4, \quad I_2 = 2p_y^2 + y^4 = \mu^4,$$

(25)

and action-angle variables may be defined everywhere in the phase space except where

$$\text{rank}\left( \frac{D(I_1, I_2)}{D(x, y, p_x, p_y)} \right) < 2,$$

that is except at the points where $\lambda$ or $\mu$ are zero. These correspond to the straight-line solutions along the two axes. The invariant tori of the system are defined by pairs of non-zero values of $\lambda$, $\mu$. The periods of $x$ and $y$ are respectively

$$T_x = 4\frac{K}{\lambda}, \quad T_y = 4\frac{K}{\mu},$$

(26)

where $K = K(1/\sqrt{2})$ is the complete elliptic integral of the first kind with modulus $k = 1/\sqrt{2}$. We can easily find out the periodic solutions of $H_0$ by demanding commensurability between $T_x$, $T_y$, i.e.

$$nT_x = mT_y, \quad m, n \in \mathbb{Z}\setminus\{0\},$$

(27)

with $m, n$ relative primes. The periodicity of $x$ and $y$ is inherited by the angles $w_1, w_2$ respectively since the system is separable and so, on a resonant torus we have

$$\omega_1 = \lambda, \quad \omega_2 = \mu,$$

where $\omega_1, \omega_2$ are the corresponding frequencies

$$\omega_1 = \frac{2\pi\lambda}{4K}, \quad \omega_2 = \frac{2\pi\mu}{4K}.$$  

(28)

The specific frequency $\Omega$ of the orbits on the resonant torus as given by (10) is

$$\Omega = \frac{\pi\lambda}{2nK} = \frac{\pi\mu}{2mK},$$

and on the particular torus we may put

$$\lambda = gn, \quad \mu = gm,$$

(29)

where

$$g = \frac{2K}{\pi} \Omega.$$  

(30)
The parameter \( g \) labels a specific torus of the family of resonant tori with frequency ratio \( n/m \). Solution (24) acquires the form

\[
\begin{align*}
x &= gn \cn(gnt - \varphi_1, 1/\sqrt{2}), \\
y &= gm \cn(gmt - \varphi_2, 1/\sqrt{2}),
\end{align*}
\]

and the common period is

\[
T = \frac{4K}{g}. \tag{31}
\]

Another important point before applying the criterion to the perturbed system is to find out the one-to-one correspondence between \( \varphi_1, \varphi_2 \) and the different orbits on the torus. The parameter which defines a particular orbit on the resonant torus is easily found to be

\[
\varphi = m\varphi_1 - n\varphi_2 \mod 4K.
\]

The perturbation \( H_1 \) is the perturbative potential

\[
V_1 = ax^2 + \beta y^2 + 2xy.
\]

If one keeps only the separable part of \( V_1 \), the resulting Hamiltonian is integrable and, as one can easily verify, the integral

\[
\int_0^T (ax^2 + \beta y^2) \, dt
\]

assumes the same value along every solution on a specific resonant torus. On the other hand, the average value of \( 2xy \) is

\[
c_0(\varphi_1, \varphi_2) = \frac{g^2 mn}{2K} \int_0^{4K/g} \cn(gnt - \varphi_1) \times \cn(gmt - \varphi_2) \, dt. \tag{32}
\]

For \( \varphi_1 = K, \varphi_2 = 0 \), which correspond to \( \varphi = mK \) (mod \( 4K \)), the integral (32) becomes

\[
c_0(K, 0) = \frac{g^2 mnk'}{2K} \int_0^{4K/g} \sn(nt) \cn(mt) \, d\tau, \tag{33}
\]

where \( k' = (1 - k^2)^{1/2} = 1/\sqrt{2} \) and \( \tau = gt \). Since the integrand is an odd function of \( \tau \)

\[
c_0(K, 0) = 0
\]

on the whole set of resonant tori, i.e., \( \forall m,n \).

Selecting now \( \varphi_1 = \varphi_2 = 0 \), we form the integral

\[
c_0(0, 0) = \frac{g^2 mn}{2K} \int_0^{4K} \cn(n\tau) \cn(m\tau) \, d\tau. \tag{34}
\]

This choice of \( \varphi_1, \varphi_2 \) corresponds to \( \varphi = 0 \), therefore, if \( m \) is a multiple of 4 we are actually dealing with the same orbit. On the other hand, if \( m \) is odd, the two integrals (33) and (34) are evaluated on different orbits of the resonant torus defined by a particular non-zero value of \( g \).

The constant coefficient of the complex Fourier expansion of the product of two even functions is given by

\[
c_0 = \frac{1}{2} \alpha_0 \alpha'_0 + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_p \alpha'_p, \tag{35}
\]

where \( \alpha_p, \alpha'_p \) are the Fourier coefficients of the cosine series of these two functions. By considering the Fourier expansion of the Jacobi elliptic cosine ([25] p. 168) we may write

\[
\begin{align*}
\cn(n\tau) &= \frac{2\pi}{kK} \sum_{r=0}^{\infty} \frac{q^{r+1/2}}{1 + q^{2r+3}} \cos \left( n(2s + 1) \frac{\pi \tau}{2K} \right), \\
\cn(m\tau) &= \frac{2\pi}{kK} \sum_{r=0}^{\infty} \frac{q^{r+1/2}}{1 + q^{2r+3}} \cos \left( m(2r + 1) \frac{\pi \tau}{2K} \right),
\end{align*}
\]

where \( q \) is the nome, which in our case is

\[
q = e^{-\pi}.
\]

In order to apply (35), we notice from (36) that the coefficients \( \alpha_p, \alpha'_p \) multiplied together correspond to

\[
p = |n|(2s + 1) = |m|(2r + 1). \tag{37}
\]

Since \( m, n \) are relative primes, they cannot be both even. When one is odd and the other even, there are no integers \( p \) satisfying eq. (37) and
E. Meletlidou, S. Ichtiaroglou / A criterion for non-integrability

267

c_0 = 0 in this case. We select therefore both m, n to be odd and relative primes and we see from (37) that for every r such that 2r + 1 is a multiple of |n| there exists only one s which satisfies (37), such that 2s + 1 is a multiple of |m|. We define an integer j by the relations

\[ 2r + 1 = |n|(2j + 1), \quad 2s + 1 = |m|(2j + 1), \]

and expression (35), after using (36), yields

\[ c_0(0,0) = \frac{8\pi^2}{K^2} \frac{g^2 mn}{\sum_{j=0}^{\infty} q^{j+1/2}(|m|+|n|)} (1 + q^{|m|(2j+1)})(1 + q^{|n|(2j+1)}). \] (38)

The infinite series in (38) is greater than zero and converges to a positive number. Relation (38) is valid for the whole family of tori with frequency ratio of the form odd/odd. On the other hand, rationals of this form are dense in \( \mathbb{R} \). Therefore we have found a dense set of tori on which the average value of \( H_1 \) is different on two orbits of the same torus. According to our proposition, this proves that the Hamiltonian (22) is non-integrable for \( \epsilon \neq 0 \) in an open interval of \( \epsilon \) around zero. This, however, means that the Hamiltonian (22) is actually non-integrable for any \( \epsilon \neq 0 \), since the corresponding equations of motion remain invariant under the transformation

\[ x \rightarrow cx, \quad y \rightarrow cy, \quad t \rightarrow c^{-1}t, \quad \epsilon \rightarrow c^2 \epsilon, \]

where \( c \) is an arbitrary parameter.

By diagonalizing the quadratic form in \( V_1 \), we also have shown non-integrability of any potential of the form

\[ V = \frac{1}{2}(Ax^2 + By^2) + C(x^4 + y^4) - Dxy(x^2 - y^2) \]

\[ + Ex^2y^2, \quad A \neq B, \]

with

\[ C : D : E = (1 + \gamma^4) : 4\gamma(1 - \gamma^2) : 12\gamma^2, \]

where the ratio \( A : B \) and the non-zero parameter \( \gamma \) are arbitrary. The above form of the potential may be applicable in problems of galactic dynamics. The criterion cannot be applied directly to this potential, considering the harmonic oscillator as the integrable part, since it is degenerate, that is relation (5) does not hold.

5. Concluding remarks

The proposition proved in section 3 of this paper is a useful tool in proving non-integrability of a two-degrees-of-freedom perturbed Hamiltonian of the form (1). According to our result, integrability of this system is related to the behaviour of the perturbing function \( H_1 \), considered as a function of time along the non-isolated periodic orbits of \( H_0 \). More specifically, if the average value of \( H_1 \), evaluated along the periodic orbits on a resonant torus of \( H_0 \) depends on the particular orbit and if this is true for a dense set of tori, then the perturbed system is non-integrable in an open interval of the perturbing parameter around zero. Although this result is a consequence of Poincaré’s theorem, its viewpoint is different. In order to apply Poincaré’s theorem for a specific perturbation one needs to transform to action-angle variables and evaluate the coefficients of the double Fourier series of \( H_1 \) with respect to the angles in an open domain of phase space and then show that an infinite number of them, one at least for each class, is different from zero at the corresponding points where eq. (7) is valid. On the other hand, for applying the present proposition, one needs to evaluate the average value of \( H_1 \) with respect to time along the periodic orbits of \( H_0 \) using any canonical variables. This is an easier task and can be met with more applications, although the knowledge of the periodic solutions on a dense set of tori is more or less equivalent to obtaining a global transformation to action-angle variables.

There is an apparent connection of our result to the theory of subharmonic bifurcations of the periodic orbits under the perturbation ([24], p. 109; see also [26], p. 117]) and our quantity...
\(\frac{d\langle H_1\rangle}{d\theta}\) is actually the Melnikov subharmonic function. Simple zeroes of this function on a resonant torus correspond to periodic orbits which can be continued analytically for \(\varepsilon \neq 0\). An attempt to prove non-integrability by using the isolateness of these orbits for \(\varepsilon \neq 0\) and the continuity of the Jacobian of \(H\) and \(\Phi\) with respect to \(\varepsilon\) is made in [27] but is demolished by the fact that one cannot prove the existence of isolated periodic orbits of arbitrarily long periods for small but fixed \(\varepsilon \neq 0\) [5,28,29]. The existence of simple zeroes of this function on a dense set of tori of \(H_0\) guarantees the existence of orbits on these tori where this function is different from zero and, according to our result, this suffices for non-integrability of \(H\) for small but fixed \(\varepsilon \neq 0\), without any consideration of the bifurcated orbits and their stability.

There are also affinities of our method to the one applied by Newcomb [30] to obtain a criterion for the solvability of the “magnetic differential equation”. Equation (4) has actually the form of such an equation and (12) is related to Newcomb’s necessary condition. This equation, however, cannot be used as a criterion in our case, since \(\Phi_0\), apart from its general properties mentioned in section 2, is not known.

We applied our result to the separable quartic oscillator with a non-separable quadratic perturbation and proved non-integrability of this system for all non-zero values of \(\varepsilon\) by taking advantage of the special form of the potential, i.e. sum of two homogeneous terms. Yoshida’s [13] criterion for such potentials or Melnikov’s homoclinic method cannot be applied in this case, since both terms are by themselves integrable and \(H_0\) does not possess a homoclinic loop. On the other hand, Ito’s [15] method can be applied only when \(\alpha = \beta\), in which case \(H\) possesses straight-line solutions.

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